**MAS 341: GRAPH THEORY**
**2016 EXAM SOLUTIONS**

## 1. Question 1

**Part i.** There are many Hamiltonian cycles; e.g., ABLKJEDFIHGC.

We now show that if we remove vertex $D$, the result $G \setminus \{D\}$ is not hamiltonian.

Note first that $G$ is trivalent, so removing $D$ makes the vertices adjacent to it, namely, $C, E$ and $F$, all have degree two. Thus, any Hamiltonian path in $G \setminus E$ would have to use $ACG$, $HFI$, $BEJ$, or their reverses.

As a result of this, we see the edge $HI$ cannot be included in a Hamiltonian cycle, as this would produce instead the three cycle $HFI$. But as $H$ and $I$ have degree 3, this means the edges $HG$, and $IJ$, must be included in any Hamiltonian cycle of $G \setminus E$. Thus, we see any Hamiltonian cycle would have to include $ACGHFIJEB$ or its reverse. But now consider vertex $K$, which is adjacent to $G, J$ and $L$. The Hamiltonian cycle already visits $G$ and $J$, and so we cannot include an edge from $K$ to either of these vertices, and so $K$ cannot be included in a Hamiltonian cycle, and hence there cannot be one.

**Part ii.** A graph $G$ is Eulerian if it has a walk that uses every edge exactly once. Euler's theorem says that a connected graph $G$ is Eulerian if and only if the degree of every vertex is even.

Suppose that $G$ is connected and every vertex of $G$ has even degree. First, note that we can construct a closed walk in $G$ that does not repeat any edges: simply start from any vertex $v$ and start walking. When our path enters a new vertex, it uses up an edge, and when it leaves a new vertex it uses up another, and so we will always use up an even number of edges; thus, whenever we arive at a new vertex $w$ there will always be a vertex to leave by unless we have returned back to $v$.

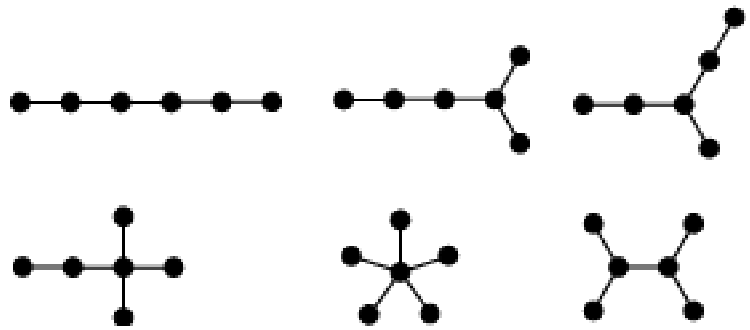
Now we can prove $G$ is Eulerian by induction on the number of edges in the graph. If $G$ has no edges, it is just a single vertex, and the theorem is trivial. Assume that $G$ has $m$ edges and all graphs with less than $m$ edges are Eulerian. From the above, $G$ has a closed walk $w$ that doesn't repeat edges. Considering the graph $G'$ obtained by deleting all the edges of $w$. Each connected component of $G'$ has vertices only of even degree, and has fewer than $m$ edges, and so has an Eulerian cycle. We can obtain an Eulerian cycle for $G$ by stitching together $w$ with the eulerian cycle for each component of $G'$

**Part iii.** By definition, a graph $G$ is semi-Eulerian if it has a (not necessarily closed) walk that uses every edge exactly once.

The analog of Euler's theorem is that a connected graph $G$ is semi-Eulerian if and only if it has at most two vertices of odd degree.

To prove this condiition is necessary, suppose $G$ had a semi-Eulerian walk $w$ that starts at $u$ and ends at $v$. Whenever a vertex appears in $W$ besides the endpoints, the walk uses up two edges; since every edge is used, every vertex except perhaps $u$ and $v$ must have even degree.

**Part iv.** There are six unlabelled trees on six vertices, shown below:

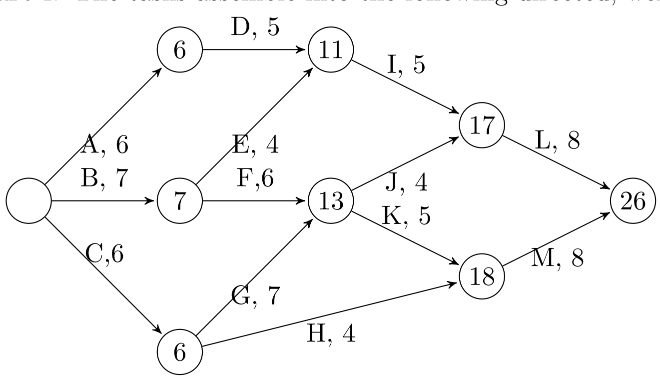


One way to be systematic is to consider the largest degree of any vertex; a different way is to consider the length of the longest path in T.

To count labelled trees, we use Cayley's theorem: there are $n^{n-2}$ labelled trees on $n$ vertices, and hence $6^4 = 1296$ labelled trees on six vertices.

## Question 2

**Part i.** The tasks assemble into the following directed, weighted graph:




The numbers in the vertices are the lengths of the longest paths to that vertex, obtained from running the longest path algorithm; thus, the project takes 26 days.

There are in fact two longest paths that take 26 days: BFKM and CGKM. If any task in any longest edge runs over, the entire project runs over, hence the six tasks BCFGKM would make the project run over.

**Part ii.** To shorten the project, we must shorten the length of an edge that's contained in EVERY longest path, hence only the two edges K and M could shorten the project. However, while KM costs 13, the path JL costs 12, so even if we shorten K or M by two days, we only shorten the length of the whole project by one day. Hence, shortening K or M shorten the entire project by 1 day, and there are no tasks for which shortening them shortens the entire project by two days.

**Part iii.** The traveling salesman problem is the following: given a weighted graph $(G, w)$, find the cheapest Hamiltonian cycle in $G$. A lower bound for the TSP may be obtained as follows: choose a vertex $v$ of $G$. Find a minimal spanning tree $T$ of $G \setminus v$ – this can be easily done using Kruskal's algorithm. Let $e$ and $f$ be the two edges out of $v$ with smallest weight. Then $w(T)+w(e)+w(f)$ is a lower bound for the TSP.

**Part iv.** To prove that this is in fact a lower bound, let

$$v = v_0 \stackrel{e_1}{\to} w_1 \stackrel{e_2}{\to} w_2 \stackrel{e_3}{\to} \cdots \stackrel{e_n}{\to} w_n = v$$

be a Hamiltonian cycle. We see that since $e_0$ and $e_n$ are two different edges incident to $v$, we have $w(e_0) + w(e_1) > w(e) + w(f)$ since $e$ and $f$ were the two cheapest edges incident to $v$. Since $w_1, \cdots, w_n$ are all the vertices of $G \setminus v$, the walk

$$w_2 \stackrel{e_2}{\to} w_3 \stackrel{e_3}{\to} \cdots \stackrel{e_{n-1}}{\to} w_{n-1}$$

is a spanning tree of $G \setminus v$, and so we have that $w(e_1)+w(e_2)+\ldots w(e_{n-1}) > w(T)$. Adding these two inequalities gives the desired result.

**Part v.** Consider a weighted graph $G$ on vertices $a, b, c, d$, where edges $ad$ and $bd$ had weight 10, and all other edges (namely $ab, ac, bc, cd$) had weight 1. Run the algorithm described by forgetting the vertex $a$. Then, the two cheapest edges out of $a$ (namely $ab$ and $ac$) both have weight 1. Meanwhile, the cheapest spanning tree of $G \setminus a$ has weight 2 (containing $bc$ and $cd$). Thus, the algorithm gives a lower bound of 4 for the TSP.

It is easy to see that this bound cannot be obtained by considering the vertex $d$ – it is incident to three edges, with weights 1, 10 and 10, and so when we visit $d$ we must use edges worth at least 11.

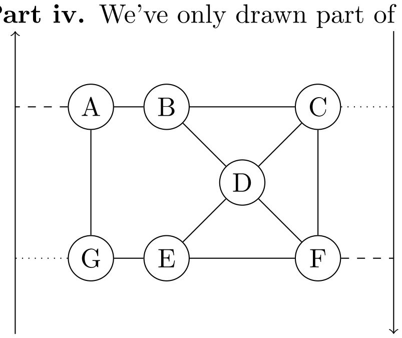
**Question 3.**

**Part i.** Kuratowski's theorem states that a graph $G$ is nonplanar if and only if it has a subgraph that's a subdivision of $K_5$ or $K_{3,3}$. If we colour $ACE$ red and $BGF$ blue, then we almost have a $K_{3,3}$: $A$ and $C$ are adjacent to all of $BFG$, and the only edge we are missing is $BE$, which can be connected through $D$.

**Part ii.** Deleting any of the 10 edges used in the above will result in a planar graph; a planar embedding of the result should be given for full justification.

**Part iii.** Deleting either of the two edges edge not used in part (i) – namely, $DC$ and $DF$ – will result in a graph that still contains a subdivision of $K_{3,3}$ as a subgraph and hence cannot be planar.

**Part iv.** We've only drawn part of the boundary of the projective plane.



**Part v.** Suppose that $G$ is drawn on the sphere with $v$ vertices, $e$ edges, $p$ pentagonal faces, and $h$ heptagonal faces, for $p+h$ total faces. Euler's theorem gives $v-e+p+h=2$. Since each vertex of $G$ has degree 3, the handshaking lemma gives $3v=2e$. Handshaking between edges and faces gives $5p+7h=2e$. Hence, $3v=2e=5p+7h$, and we may eliminate $v$ and $e$ from Euler's theorem to obtain an equation between $p$ and $h$; the slickest way is to multiply euler's equation by 6 to get $6v-6e+6p+6h=12$, But $6v-6e=-5p-7h$, which gives $p-h=12$ as desired.

## Question 4

**Part i.** The chromatic polynomial $\chi_G(k)$ counts the number of ways to colour every vertex of $G$ with one of $k$ different colours, with the requirement that no two adjacent vertices have the same colour. If $n$ is the number of vertices of $G$ and $m$ is the number of edges of $G$, then $\chi_G(k)=k^n-mk^{n-1}+\cdots$; in other words, $n$ is the degree of $\chi_G(k)$, and $-m$ is the coefficient of $k^{n-1}$.

**Part ii.** If $e$ is any edge of $G$, let $G/e$ be the graph obtained by contracting $e$, i.e., shrinking $e$ to a point and making it a vertex, and let $G\setminus e$ be the graph obtained by deleting $e$. Then the deletion-contraction equation says that $\chi_G(k)=\chi_{G\setminus e}(k)-\chi_{G/e}(k)$.

To prove this, first note that any colouring of $G$ is a colouring of $G\setminus e$. A colouring of $G\setminus e$ gives a colouring of $G$ if and only if the two vertices of $e$ have different colours. Meanwhile, a colouring of $G\setminus e$ gives a colouring of $G/e$ if and only if the two vertices of $e$ have the *same* colour. As in any colouring of $G\setminus e$ the two vertices of $e$ must are either the same or different, this gives $\chi_{G\setminus e}(k)=\chi_{G/e}(k)+\chi_G(k)$, which is equivalent to the desire result.

**Part iii.** The proof is nested induction on the number of vertices $n$ of $G$ and the number of edges $m$ of $G$. As a base case, consider the empty graph $E_n$, which has $n$ vertices and no edges. Since each vertex of $E_n$ can be coloured any of the $k$ colours independently, we see $\chi_{E_n}(k)=k^n$ is a polynomial as desired.

Now, assume that for any graph $\Gamma$ with less than $n$ vertices, or with exactly $n$ vertices and less than $m$ edges, we know that $\chi_\Gamma(k)$ is a polynomial, and let $G$ be a graph with $n$ vertices and $m$ edges. We may assume $G$ has at least one edge $e$, or we have the empty graph, which we have covered. Applying deletion contract to the edge $e$, we see

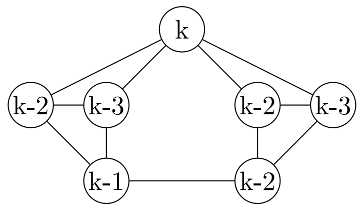
$$\chi_G(k)=\chi_{G\setminus e}(k)-\chi_{G/e}(k)$$

But $G\setminus e$ has the same number of vertices but fewer edges than $G$, and so we know $\chi_G(k)$ is a polynomial by the inductive hypothesis. Similarly, $G/e$ has fewer vertices than $G$ and so $\chi_G(k)$ is a polynomial. Thus, $\chi_G(k)$ is the difference of two polynomials, and hence a polynomial.

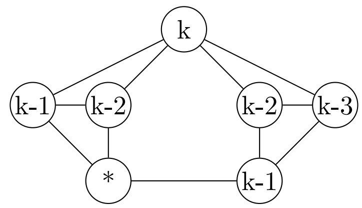
**Part iv.**

*Bare-bones approach:* Consider the vertices $A, F$ and $G$. $F$ and $G$ are adjacent, and so most be different colours; we thus have two cases – either $A, F$ and $G$ are three different colours, of $A$ is the same colour as either $F$ or $G$.

Colouring Case 1




Colouring Case 2




In case 1 then, there are $k(k-1)(k-2)$ ways to colour $A, F$ and $G$. We see there are $(k-2)(k-3)$ ways to colour $B$ and $C$, and also to colour $D$ and $E$, and so in total case 1 contains $k(k-1)(k-2)^3(k-3)^2$ colourings.

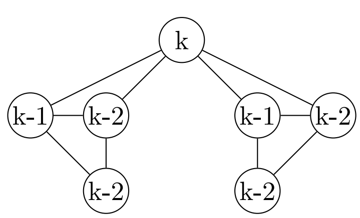
For Case 2, assume that $F$ and $A$ are the same colour. Then there are $k(k-1)$ ways to colour $A, F$ and $G$. There are $k-1$ ways to colour $B$ and $k-2$ ways to colour $C$; then $k-2$ ways to colour $D$ and $k-3$ ways to colour $E$. This gives $k(k-1)^2(k-2)^2(k-3)$ colourings when $F$ and $A$ are the same colour; by symmetry, there are just as many colourings when $A$ and $G$ have the same colour. Thus, in total we have

$$
\begin{aligned}
\chi_G(k) &= k(k-1)(k-2)^3(k-3)^2 + 2k(k-1)^2(k-2)^2(k-3) \\
&= k(k-1)(k-2)^2(k-3)[(k-2)(k-3) + 2(k-1)] \\
&= k(k-1)(k-2)^2(k-3)[k^2 - 3k + 4]
\end{aligned}
$$

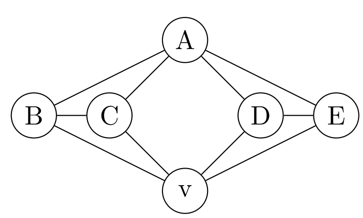
*Using Deletion Contraction.* Alternatively, we may first apply deletion contraction to the edge $e = FG$.

If we delete it, a quick argument gives $\chi_{G\setminus e}(k(k-1)^2(k-2)^4$.

Colouring $H \setminus FG$

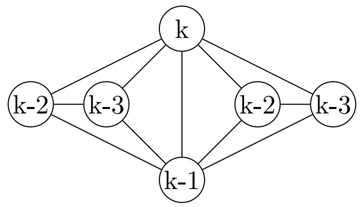



$H/FG$

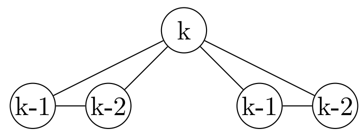



It is a little more difficult to calculate $\chi_{G/e}(k)$. Let $v \in G/e$ be the vertex new that came from contracting $F - G$. We see that adding the edge $G/e \cup Av$ gives a graph whose polynomial is easily found by first colouring $A$ then colouring $v$ – we have $\chi_{G/e\cup Av} = k(k-1)(k-2)^2(k-3)^2$. If we then contract this new edge, to get the new graph $\Gamma = (G/e \cup Av)/Av$ has some multiple edges that can be deleted, and it is easy to calculate $\chi_\Gamma(k) = k(k-1)^2(k-2)^2$.

Colouring $H \setminus FG \cup Av$




Colouring $\Gamma$




So,

$$\begin{aligned}
\chi_G(k) &= \chi_{G\setminus e}(k) - \chi_{G/e}(k) \\
&= \chi_{G\setminus e}(k) - \chi_{G/e\cup Av}(k) - \chi_\Gamma(k) \\
&= k(k-1)^2(k-2)^4 - k(k-1)(k-2)^2(k-3)^2 - k(k-1)^2(k-2)^2 \\
&= k(k-1)(k-2)^2[(k-1)(k-2)^2 - (k-3)^2 - (k-1)] \\
&= k(k-1)(k-2)^2[(k-1)(k^2-4k+3) - (k-3)^2] \\
&= k(k-1)(k-2)^2[(k-3)(k-1)^2 - (k-3)^2] \\
&= k(k-1)(k-2)^2(k-3)[k^2-2k+1-k+3] \\
&= k(k-1)(k-2)^2(k-3)(k^2-3k+4)
\end{aligned}$$

**Part v.** Since $A$ has degree 4, we see that at least 4 colours are necessary. We show 4 colours are possible as follows: colour $AB, CF$ and $DE$ red, colour $AC, BF$ and $EG$ blue, colour $AE, BC$ and $DG$ yellow, and colour $AD$ and $FG$ green.