1. Question 1

Part i. There are many Hamiltonian cycles; e.g., ABLKJEDFIHGC.

We now show that if we remove vertex $D$, the result $G \setminus \{D\}$ is not hamiltonian.

Note first that $G$ is trivalent, so removing $D$ makes the vertices adjacent to it, namely, $C, E$ and $F$, all have degree two. Thus, any Hamiltonian path in $G \setminus E$ would have to use $ACG, HFI, BEJ$, or their reverses.

As a result of this, we see the edge $HI$ cannot be included in a Hamiltonian cycle, as this would produce instead the three cycle $HFI$. But as $H$ and $I$ have degree 3, this means the edges $HG$, and $IJ$, must be included in any Hamiltonian cycle of $G \setminus E$. Thus, we see any Hamiltonian cycle would have to include $ACGHFIJEB$ or its reverse. But now consider vertex $K$, which is adjacent to $G, J$ and $L$. The Hamiltonian cycle already visits $G$ and $J$, and so we cannot include an edge from $K$ to either of these vertices, and so $K$ cannot be included in a Hamiltonian cycle, and hence there cannot be one.

Part ii. A graph $G$ is Eulerian if it has a walk that uses every edge exactly once. Euler’s theorem says that a connected graph $G$ is Eulerian if and only if the degree of every vertex is even.

Suppose that $G$ is connected and every vertex of $G$ has even degree. First, note that we can construct a closed walk in $G$ that does not repeat any edges: simply start from any vertex $v$ and start walking. When our path enters a new vertex, it uses up an edge, and when it leaves a new vertex it uses up another, and so we will always use up an even number of edges; thus, whenever we arrive at a new vertex $w$ there will always be a vertex to leave by unless we have returned back to $v$.

Now we can prove $G$ is Eulerian by induction on the number of edges in the graph. If $G$ has no edges, it is just a single vertex, and the theorem is trivial. Assume that $G$ has $m$ edges and all graphs with less than $m$ edges are Eulerian. From the above, $G$ has a closed walk $w$ that doesn’t repeat edges. Considering the graph $G'$ obtained by deleting all the edges of $w$. Each connected component of $G'$ has vertices only of even degree, and has fewer than $m$ edges, and so has an Eulerian cycle. We can obtain an Eulerian cycle for $G$ by stitching together $w$ with the eulerian cycle for each component of $G'$

Part iii. By definition, a graph $G$ is semi-Eulerian if it has a (not necessarily closed) walk that uses every edge exactly once.

The analog of Euler’s theorem is that a connected graph $G$ is semi-Eulerian if and only if it has at most two vertices of odd degree.

To prove this condition is necessary, suppose $G$ had a semi-Eulerian walk $w$ that starts at $u$ and ends at $v$. Whenever a vertex appears in $W$ besides the endpoints, the walk uses up two edges; since every edge is used, every vertex except perhaps $u$ and $v$ must have even degree.
Part iv. There are six unlabelled trees on six vertices, shown below:

One way to be systematic is to consider the largest degree of any vertex; a different way is to consider the length of the longest path in $T$.

To count labelled trees, we use Cayley’s theorem: there are $n^{n-2}$ labelled trees on $n$ vertices, and hence $6^4 = 1296$ labelled trees on six vertices.

Question 2

Part i. The tasks assemble into the following directed, weighted graph:

The numbers in the vertices are the lengths of the longest paths to that vertex, obtained from running the longest path algorithm; thus, the project takes 26 days.

There are in fact two longest paths that take 26 days: BFKM and CGKM. If any task in any longest edge runs over, the entire project runs over, hence the six tasks BCFGKM would make the project run over.

Part ii. To shorten the project, we must shorten the length of an edge that’s contained in EVERY longest path, hence only the two edges K and M could shorten the project. However, while KM costs 13, the path JL costs 12, so even if we shorten K or M by two days, we only shorten the length of the whole project by one day. Hence, shortening K or M shorten the entire project by 1 day, and there are no tasks for which shortening them shortens the entire project by two days.

Part iii. The traveling salesman problem is the following: given a weighted graph $(G, w)$, find the cheapest Hamiltonian cycle in $G$. A lower bound for the TSP may be obtained as follows: choose a vertex $v$ of $G$. Find a minimal spanning tree $T$ of $G \setminus v$ – this can be easily done using Kruskal’s algorithm. Let $e$ and $f$ be the two
edges out of \( v \) with smallest weight. Then \( w(T) + w(e) + w(f) \) is a lower bound for the TSP.

**Part iv.** To prove that this is in fact a lower bound, let
\[
v = v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \xrightarrow{e_3} \cdots \xrightarrow{e_n} v_n = v
\]
be a Hamiltonian cycle. We see that since \( e_0 \) and \( e_n \) are two different edges incident to \( v \), we have \( w(e_0) + w(e_1) > w(e) + w(f) \) since \( e \) and \( f \) were the two cheapest edges incident to \( v \). Since \( v_1, \ldots, v_n \) are all the vertices of \( G \setminus v \), the walk
\[
w_2 \xrightarrow{e_3} w_3 \xrightarrow{e_4} \cdots \xrightarrow{e_{n-1}} w_{n-1}
\]
is a spanning tree of \( G \setminus v \), and so we have that \( w(e_1) + w(e_2) + \ldots w(e_{n-1}) > w(T) \). Adding these two inequalities gives the desired result.

**Part v.** Consider a weighted graph \( G \) on vertices \( a, b, c, d \), where edges \( ad \) and \( bd \) had weight 10, and all other edges (namely \( ab, ac, bc, cd \)) had weight 1. Run the algorithm described by forgetting the vertex \( a \). Then, the two cheapest edges out of \( a \) (namely \( ab \) and \( ac \)) both have weight 1. Meanwhile, the cheapest spanning tree of \( G \setminus a \) has weight 2 (containing \( bc \) and \( cd \)). Thus, the algorithm gives a lower bound of 4 for the TSP.

It is easy to see that this bound cannot be obtained by considering the vertex \( d \) – it is incident to three edges, with weights 1, 10 and 10, and so when we visit \( d \) we must use edges worth at least 11.

**Question 3.**

**Part i.** Kuratowski’s theorem states that a graph \( G \) is nonplanar if and only if it has a subgraph that’s a subdivision of \( K_5 \) or \( K_{3,3} \). If we colour \( ACE \) red and \( BGF \) blue, then we almost have a \( K_{3,3} \): \( A \) and \( C \) are adjacent to all of \( BFG \), and the only edge we are missing is \( BE \), which can be connected through \( D \).

**Part ii.** Deleting any of the 10 edges used in the above will result in a planar graph; a planar embedding of the result should be given for full justification.

**Part iii.** Deleting either of the two edges edge not used in part (i) – namely, \( DC \) and \( DF \) – will result in a graph that still contains a subdivision of \( K_{3,3} \) as a subgraph and hence cannot be planar.

**Part iv.** We’ve only drawn part of the boundary of the projective plane.
Part v. Suppose that $G$ is drawn on the sphere with $v$ vertices, $e$ edges, $p$ pentagonal faces, and $h$ heptagonal faces, for $p + h$ total faces. Euler’s theorem gives $v - e + p + h = 2$. Since each vertex of $G$ has degree 3, the handshaking lemma gives $3v = 2e$. Handshaking between edges and faces gives $5p + 7h = 2e$. Hence, $3v = 2e = 5p + 7h$, and we may eliminate $v$ and $e$ from Euler’s theorem to obtain an equation between $p$ and $h$; the slickest way is to multiply Euler’s equation by 6 to get $6v - 6e + 6p + 6h = 12$, But $6v - 6e = -5p - 7h$, which gives $p - h = 12$ as desired.

Question 4

Part i. The chromatic polynomial $\chi_G(k)$ counts the number of ways to colour every vertex of $G$ with one of $k$ different colours, with the requirement that no two adjacent vertices have the same colour. If $n$ is the number of vertices of $G$ and $m$ is the number of edges of $G$, then $\chi_G(k) = k^n - mk^{n-1} + \cdots$; in other words, $n$ is the degree of $\chi_G(k)$, and $-m$ is the coefficient of $k^{n-1}$.

Part ii. If $e$ is any edge of $G$, let $G/e$ be the graph obtained by contracting $e$, i.e., shrinking $e$ to a point and making it a vertex, and let $G \setminus e$ be the graph obtained by deleting $e$. Then the deletion-contraction equation says that $\chi_G(k) = \chi_{G\setminus e}(k) - \chi_{G/e}(k)$.

To prove this, first note that any colouring of $G$ is a colouring of $G\setminus e$. A colouring of $G \setminus e$ gives a colouring of $G$ if and only if the two vertices of $e$ have different colours. Meanwhile, a colouring of $G \setminus e$ gives a colouring of $G/e$ if and only if the two vertices of $e$ have the *same* colour. As in any colouring of $G \setminus e$ the two vertices of $e$ must be either the same or different, this gives $\chi_{G\setminus e}(k) = \chi_{G/e}(k) + \chi_G(k)$, which is equivalent to the desired result.

Part iii. The proof is nested induction on the number of vertices $n$ of $G$ and the number of edges $m$ of $G$. As a base case, consider the empty graph $E_n$, which has $n$ vertices and no edges. Since each vertex of $E_n$ can be coloured any of the $k$ colours independently, we see $\chi_{E_n}(k) = k^n$ is a polynomial as desired.

Now, assume that for any graph $\Gamma$ with less than $n$ vertices, or with exactly $n$ vertices and less than $m$ edges, we know that $\chi_{\Gamma}(k)$ is a polynomial, and let $G$ be a graph with $n$ vertices and $m$ edges. We may assume $G$ has at least one edge $e$, or we have the empty graph, which we have covered. Applying deletion contract to the edge $e$, we see $\chi_G(k) = \chi_{G\setminus e}(k) - \chi_{G/e}(k)$.

But $G \setminus e$ has the same number of vertices but fewer edges than $G$, and so we know $\chi_G(k)$ is a polynomial by the inductive hypothesis. Similarly, $G/e$ has fewer vertices than $G$ and so $\chi_G(k)$ is a polynomial. Thus, $\chi_G(k)$ is the difference of two polynomials, and hence a polynomial.

Part iv.

Bare-bones approach: Consider the vertices $A, F$ and $G$. $F$ and $G$ are adjacent, and so must be different colours; we thus have two cases – either $A, F$ and $G$ are three different colours, of $A$ is the same colour as either $F$ or $G$.
In case 1 then, there are $k(k-1)(k-2)$ ways to colour $A,F$ and $G$. We see there are $(k-2)(k-3)$ ways to colour $B$ and $C$, and also to colour $D$ and $E$, and so in total case 1 contains $k(k-1)(k-2)^3(k-3)^2$ colourings.

For Case 2, assume that $F$ and $A$ are the same colour. Then there are $k(k-1)$ ways to colour $A,F$ and $G$. There are $k-1$ ways to colour $B$ and $k-2$ ways to colour $C$; then $k-2$ ways to colour $D$ and $k-3$ ways to colour $E$. This gives $k(k-1)^2(k-2)^2(k-3)$ colourings when $F$ and $A$ are the same colour; by symmetry, there are just as many colourings when $A$ and $G$ have the same colour. Thus, in total we have

$$\chi_G(k) = k(k-1)(k-2)^3(k-3)^2 + 2(k-1)^2(k-2)^2(k-3)$$
$$= k(k-1)(k-2)^2(k-3)[(k-2)(k-3) + 2(k-1)]$$
$$= k(k-1)(k-2)^2(k-3)[k^2 - 3k + 4]$$

Using Deletion Contraction. Alternatively, we may first apply deletion contraction to the edge $e = FG$.

If we delete it, a quick argument gives \(\chi_{G \setminus e}(k(k-1)^2(k-2)^4\)

Colouring $H \setminus FG$  

It is a little more difficult to calculate \(\chi_{G/e}(k)\). Let \(v \in G/e\) be the vertex new that came from contracting $F - G$. We see that adding the edge $G/e \cup A v$ gives a graph whose polynomial is easily found by first colouring $A$ then colouring $v$ – we have $\chi_{G/e \cup A v} = k(k-1)(k-2)^2(k-3)^2$. If we then contract this new edge, to get the new graph $\Gamma = (G/e \cup A v)/A v$ has some multiple edges that can be deleted, and it is easy to calculate $\chi_{\Gamma}(k) = k(k-1)^2(k-2)^2$. 

\[k\]
\[k-2\]
\[k-3\]
\[k-1\]
\[H/FG\]

\[k\]
\[k-1\]
\[k-2\]
\[k-3\]
\[k-1\]
\[k-2\]
\[k-3\]
\[v\]

\[A\]
\[B\]
\[C\]
\[D\]
\[E\]
Colouring $H \setminus FG \cup A\nu$

So,

\[
\chi_G(k) = \chi_{G\setminus e}(k) - \chi_{G/e}(k)
= \chi_{G\setminus e}(k) - \chi_{G/e \cup A\nu}(k) - \chi_{\Gamma}(k)
= k(k - 1)(k - 2)^2 - k^2(k - 1)(k - 2)^2(k - 3) - k(k - 1)(k - 2)^2
= k(k - 1)(k - 2)^2[(k - 1)(k - 2)^2 - (k - 3)^2 - (k - 1)]
= k(k - 1)(k - 2)^2[(k - 1)(k^2 - 4k + 3) - (k - 3)^2]
= k(k - 1)(k - 2)^2(k - 3)[k^2 - 2k + 1 - k + 3]
= k(k - 1)(k - 2)^2(k - 3)(k^2 - 3k + 1)
\]

**Part v.** Since $A$ has degree 4, we see that at least 4 colours are necessary. We show 4 colours are possible as follows: colour $AB, CF$ and $DE$ red, colour $AC, BF$ and $EG$ blue, colour $AE, BC$ and $DG$ yellow, and colour $AD$ and $FG$ green.