The University Of Sheffield.

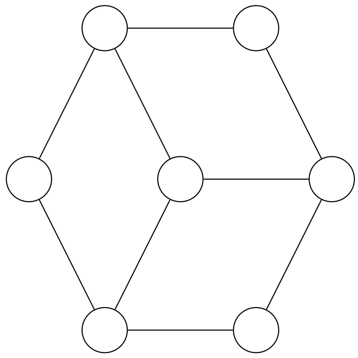
SCHOOL OF MATHEMATICS AND STATISTICS Spring Semester 2015–2016

Graph Theory 2 hours 30 minutes

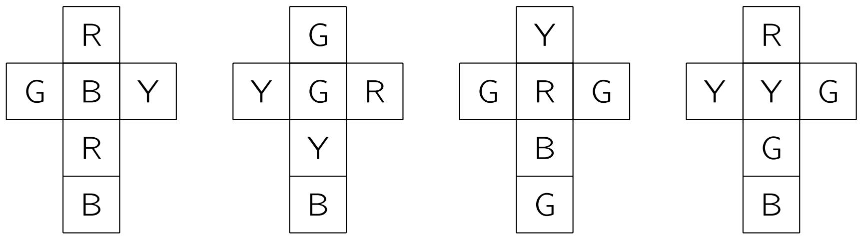
Answer FOUR questions. You are advised NOT to answer more than four questions: if you do, only your best four will be counted.

1 (i) Explain why any alkane C_nH_{2n+2} is a tree. How many isomers does C_6H_{14} have? Draw the structure of the carbon atoms in each isomer. ***(5 marks)***

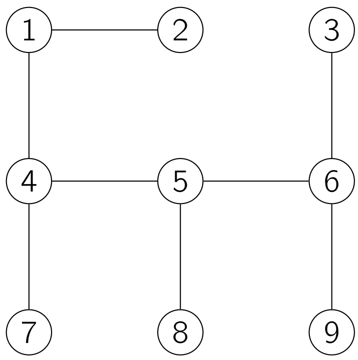
(ii) Consider the graph G given below. Is G Eulerian? Is G Hamiltonian? Is G bipartite? Justify your answers. ***(6 marks)***



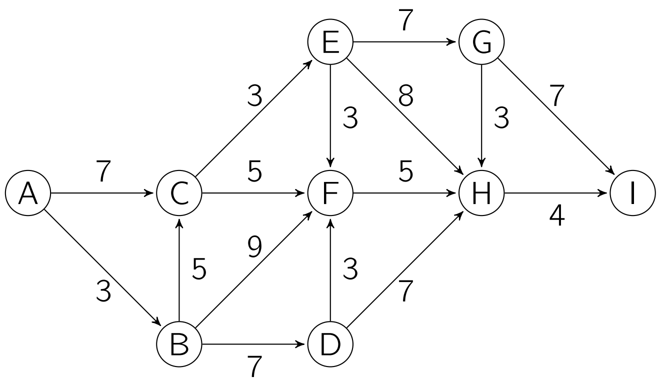
(iii) Prove that the following set of instant insanity cubes have no solution. ***(9 marks)***



(iv) Give the Prüfer code for the labelled tree T below. ***(5 marks)***



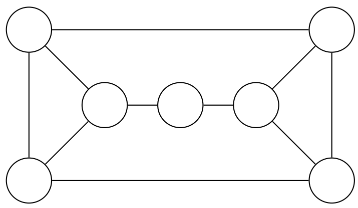
2 Consider the following directed, weighted graph:



(i) What is the length s of the shortest path from A to I? For which edges e will shortening e by 0.1 change s? For which edges e will making e longer by 0.1 change s? ***(8 marks)***

(ii) What is the length ℓ of the longest path from A to I? For which edges e will shortening e by 0.1 change ℓ? For which edges e will making e longer by 0.1 change ℓ? ***(8 marks)***

(iii) The graph Γ is shown below. Find the chromatic number and the chromatic index of Γ. ***(5 marks)***



(iv) A tree T has one vertex v of degree 4, and another vertex w of degree 3. Prove that T has at least 5 leaves. ***(4 marks)***

3 (i) Weights are given for edges between 7 vertices, labelled $A - G$.

A						
11	B					
17	9	C				
17	12	14	D			
11	17	15	10	E		
16	9	9	10	8	F	
20	10	21	19	8	12	G

Find a minimal weight spanning tree. What is the total weight of this spanning tree? ***(5 marks)***

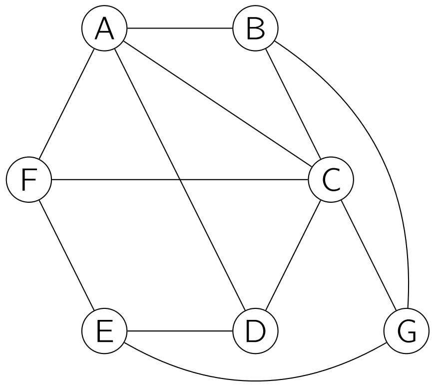
(ii) In total, how many spanning trees have the same minimum weight? ***(4 marks)***

(iii) Now, suppose the vertices represent towns, and the weights represent the cost of traveling between towns. A traveling salesperson lives in an 8th town, H. The cost of traveling from H to any town other is 25. The traveling salesperson wants to start at H, travel to every town $A - G$ exactly once, and then return to H, as cheaply as possible. Using your result from the previous part, give a lower bound on the cost of the traveling salesperson's trip. Is this lower bound attainable? Explain. ***(5 marks)***

(iv) Using the nearest neighbour heuristic, starting at H and traveling first to G, give an upper bound on the cost of the cheapest trip for the traveling salesperson. ***(3 marks)***

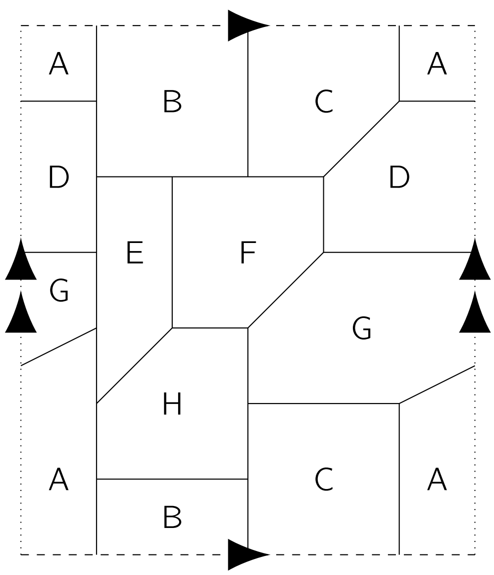
(v) Draw the Petersen graph P, and prove that P is not Hamiltonian. (Hint: Suppose that P is Hamiltonian, and consider the edges *not* in the Hamiltonian cycle) ***(8 marks)***

4 (i) State Kuratowski's theorem, and use it to show that the graph G below is not planar. Draw G on the projective plane without edges crossing. Your drawing should use the labelling of the vertices given. ***(10 marks)***

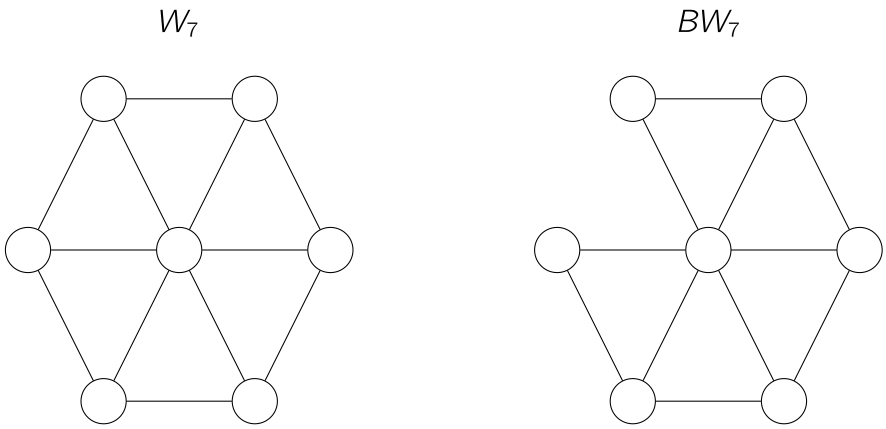


(ii) Define the Euler characteristic $\chi(S)$ of a closed, compact surface S and prove it is well defined. Use your drawing of G from Part (i) to calculate the Euler characteristic of the projective plane. ***(11 marks)***

(iii) Consider the graph Γ drawn below on the torus, with its faces labeled A through H. Give a colouring of the faces of Γ with four colours so that faces meeting along an edge have different colours. Prove that no such colouring is possible with only three colours. ***(4 marks)***



5 Recall that the wheel graph W_n consists of a copy of the cycle graph C_{n-1}, together with a central vertex v adjacent to every other vertex. Let BW_n denote the "broken wheel" graph, which is obtained from the wheel graph W_n by removing one edge from the outer cycle. We have drawn W_7 and BW_7 below:



(i) Prove, directly from the definition, that the chromatic polynomials of W_n and C_n satisfy the identity:

$$P_{W_n}(k) = kP_{C_{n-1}}(k-1)$$

(5 marks)

(ii) Prove that

$$P_{BW_n}(k) = k(k-1)(k-2)^{n-2}$$

(5 marks)

(iii) Prove that

$$P_{W_n}(k) = P_{BW_n}(k) - P_{W_{n-1}}(k)$$

Using this, the previous part, and induction, prove that

$$P_{W_n} = k(k-2)\left[(k-2)^{n-2} + (-1)^{n-1}\right]$$

(10 marks)

(iv) A graph G has chromatic polynomial $P_G(k) = k^4 - 4k^3 + 5k^2 - 2k$. How many vertices and edges does G have? Is G bipartite? Justify your answers.

(5 marks)

End of Question Paper