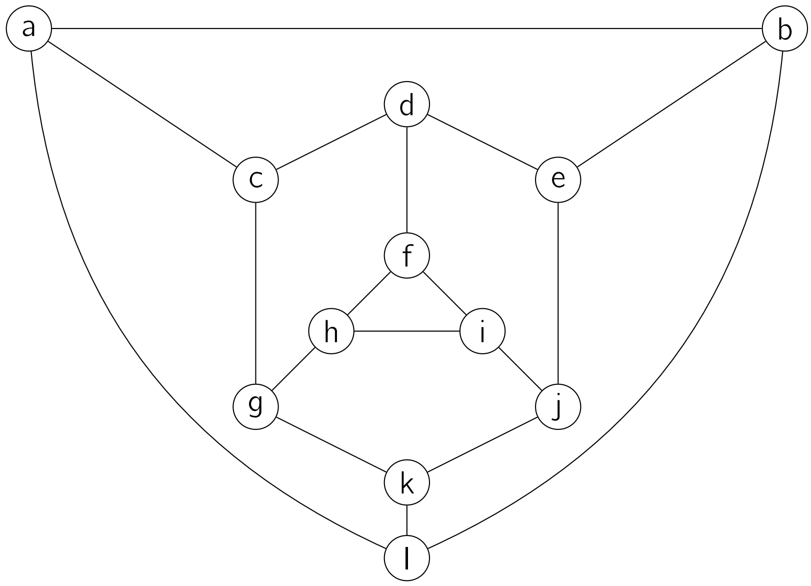
SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2016–2017

Graph Theory — **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Prove that the graph G shown below is Hamiltonian, but that if the vertex marked d is deleted, the resulting graph $G \setminus \{d\}$ is not Hamiltonian. ***(7 marks)***



(ii) State what it means for a graph G to be Eulerian. State Euler's theorem (which gives a simple, necessary and sufficient condition for a graph to be Eulerian). Prove the hard half of Euler's theorem (in other words, that if G satisfies your condition, then G is Eulerian). ***(6 marks)***

(iii) State what it means for a graph G to be semi-Eulerian. State the analogue of Euler's theorem for semi-Eulerian graphs. Prove the easy half of this theorem: namely, that if G is semi-Eulerian, then G satisfies your condition. ***(4 marks)***

(iv) How many non-isomorphic trees with six vertices are there? Draw them. How many labelled trees with six vertices are there? Don't draw them – there are too many. ***(8 marks)***

2 A large project is broken up into many small subtasks, each of which takes a given time to complete. Furthermore, any given task may have some prerequisite tasks that must be completed before the given task can be started. Assume the team is large enough that any number of tasks may be worked on at the same time.

The table below lists the tasks, the time in days needed to complete each task, and the necessary prerequisite tasks for each task.

Task	Time	Prerequisites	Task	Time	Prerequisites
A	6		H	4	C
B	7		I	5	D, E
C	6		J	4	F, G
D	5	A	K	5	F, G
E	4	B	L	8	I,J
F	6	B	M	8	H, K
G	7	C			

(i) What's the minimum number of days necessary to complete the entire project? Which tasks, if they took a day longer, would make the entire project take a day longer? Justify your answers. ***(5 marks)***

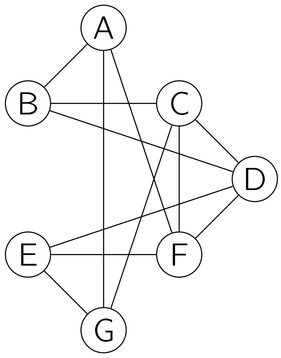
(ii) Suppose you could shorten any one subtask by two days. For which tasks would this shorten the entire project by two days? For which tasks would this shorten the entire project by one day? ***(5 marks)***

(iii) State the Travelling Salesman Problem (TSP). Describe the method we gave in lecture to construct a lower bound for the TSP. ***(5 marks)***

(iv) Prove that the method you described in Part (iii) actually produces a lower bound to the TSP. ***(5 marks)***

(v) Construct an example of a graph with four vertices that shows that the lower bound produced by the method in Part (iii) is not always achievable by a solution to the TSP. ***(5 marks)***

3 Consider the graph Γ shown here:



(i) State Kuratowski's theorem and use it to show that Γ is not planar. ***(6 marks)***

(ii) Find an edge e of Γ so that $\Gamma \setminus e$ is planar. Justify your answer. ***(4 marks)***

(iii) Find an edge f of Γ so that $\Gamma \setminus f$ is not planar. Justify your answer. ***(4 marks)***

(iv) Draw Γ on the projective plane so that no edges cross. Be sure to label the vertices of Γ. ***(4 marks)***

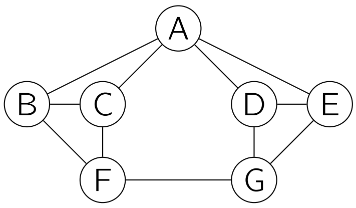
(v) A trivalent graph G is drawn on the sphere so that every face is a pentagon or a heptagon (i.e., every face has 5 or 7 sides). Prove that there are exactly 12 more pentagons than heptagons. ***(7 marks)***

4 (i) Define the chromatic polynomial $\chi_G(k)$ of a graph G. Suppose I have a simple graph G, and I tell you the chromatic polynomial $\chi_G(k)$. How can you determine how many vertices and edges my graph G has? ***(5 marks)***

(ii) State and prove the deletion-contraction equation for the chromatic polynomial. ***(5 marks)***

(iii) Prove that $\chi_G(k)$ is a polynomial function of k. You may use the deletion-contraction equation from Part (ii) even if you didn't prove it. ***(5 marks)***

The next two parts use the graph H shown here:



(iv) What is the chromatic polynomial $\chi_H(k)$ of the graph H shown above? ***(5 marks)***

(v) What is the chromatic index $\chi'(H)$ of the graph H shown above? ***(5 marks)***

End of Question Paper