Euler's Theorem: What's wrong with this picture?

Theorem: Every football has 12 pentagons

We will prove this theorem today as a corollary to Euler's Theorem.

Definition

Suppose that a connected graph G is drawn on the sphere S^2 so that no edges cross. Then G cuts the sphere into a finite number of disks called the *faces* of G.

Intuition / origin of name:

Think about the cube (or more generally a polyhedron).

Vertices	Corners
Edges	Where two cube faces meet
Faces	Faces of cube

Counting vertices, faces and edges

Graph	V	E	F
C_n	n	n	2
W_n	$n+1$	$2n$	$n+1$
$K_4 =$ Tetrahedra	4	6	4
Cube	8	12	6
Octahedron	6	12	8
Dodecahedron	20	30	12
Icosahedron	12	30	20
Make your own			

What patterns do you see?

Euler's Formula

Theorem (Euler's formula for graphs on the sphere)

Let G be a connected graph drawn on the sphere without edges crossing. Let V and E be the number of edges and vertices of G, and let F be the number of faces of the drawing. Then

$$V - E + F = 2$$

Cultural remarks

- Imre Lakatos's *Proof and Refutations* important work in philosophy of math tracing this theorem
- Beginings of topology: $V - E + F$ is called the *Euler characteristic*

Proof idea:

Deleting an edge doesn't change Euler Characteristic. Induct.

How to use Euler's Theorem

One shortcoming:

Only one equation, but three variables: V, E, F

Handshaking can give us another relation:

On a football, every vertex has degree three.

$$2E = \sum_{v \in V(G)} d(w) = \sum_{v \in V(G)} 3 = 3V$$

Similar handshaking between faces and edges

Let the *degree* $d(f)$ of a face f be the number of edges around it.

- Then each edge meets two faces
- Each face f meets $d(f)$ edges

$$\sum_{f \text{ face}} d(f) = 2E$$

Proof that a football has 12 pentagons

Definition

A *football*, we mean a graph G drawn on the plane where:

- Every vertex $v \in G$ has degree 3
- Every face f has degree 5 or 6

Suppose there are P pentagons and H hexagons, so $F = P + H$.

Basic recipe for applying Euler's theorem:

Combine the following ingredients and stir well:

- Euler's Theorem: $V - E + P + H = 2$
- Vertex-edge Handshaking: $3V = 2E$
- Face-edge Handshaking: $5P + 6H = 2E$

Back to videogames

Recall that the standard overhead view of a planet in video game produces not the sphere but the torus.

Definition

A *video game graph* is a graph drawn on a surface so that

- Every vertex has degree 4
- Every face has degree 4

Theorem

A video-game graph can never be the sphere. In fact, a video-game graph will always be the torus or the Klein bottle.

So the video-game designers didn't "mess up".

Proof: collect the standard three ingredients

Ingredient 1: Euler's theorem

Suppose that G was a video game graph drawn on the sphere:

$$V - E + F = 2$$

Ingredient 2: Vertex-edge handshaking

Since every vertex has degree 4, we have

$$2E = 4V$$

Ingredient 3: Vertex-face handshaking

Since every face has degree 4, we have

$$2E = 4F$$

Mix well to finish proof...