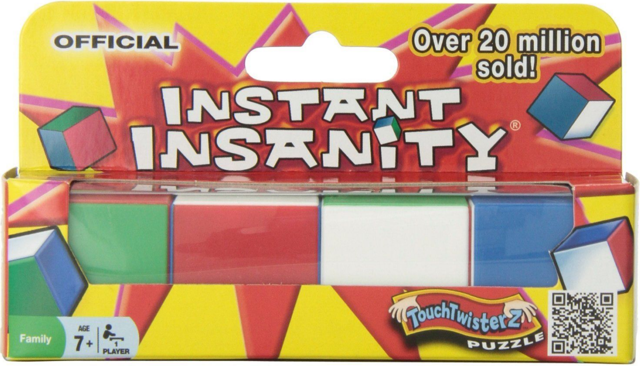
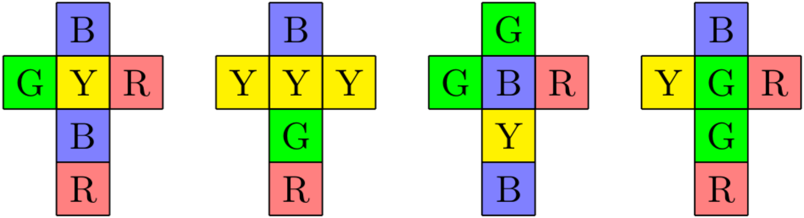
Instant Insanity / The Four Cubes Problem



- Four cubes
- Each face coloured one of four colours: blue, green, red, yellow
- Arrange the cubes in a line so that each of the four long rows has one of each colour

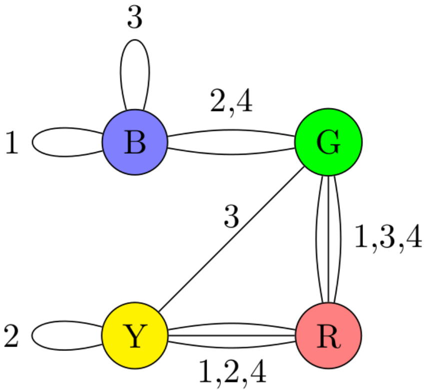
Prove the following set of cubes has no solution:



Not clear how to use graph theory....

...make a graph that encodes which faces are *opposite* each other on each cube.

The four cubes encoded in a labelled graph



What would a solution look like in terms of the graph?

Consider front/back of cube first...

A solution gives a certain pair of subgraphs...

Looking at the edges encoding front/back of cubes

1. One edge with each label in $\{1, 2, 3, 4\}$
2. Each vertex has degree two

Exactly the same with the edges encoding top/bottom of cubes.

Key fact:

A solution to Instantity Insanity gives a pair of subgraphs satisfying 1+2, and the subgraphs have with *disjoint edges*.

Can also go from a pair of subgraphs to a solution

But slightly subtly, there might be several ways to do this: for each edge, need to *choose* which colour is in front and which is in back...

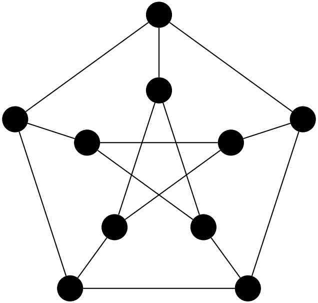
Basic graphs and concepts

- The *empty graph* E_n has n vertices and no edges
- The *complete graph* K_n has n vertices, and each vertex is connected to every other.
- The *path graph* P_n has n vertices $\{1, \ldots, n\}$ with an edge between i and $i+1$
- The *cycle graph* C_n has n vertices $\{1, \ldots, n\}$ with an edge between i and $i+1$ and between n and 1.

Definition

Let G be a simple graph with vertex set V. Its *complement* G^c is another graph with vertex set V, where two vertices $v, w \in V$ are adjacent in G^c if and only if they are not adjacent in G.

Obligatory Petersen graph



What does it mean for a graph to be connected?

Connected means we can "get from any vertex to another"

Definition (Walk)

Let G be a simple graph. A *walk* in G is a sequence of vertices $v_1, v_2, \ldots, v_n$ so that v_i is adjacent to v_{i+1}. We we say the walk goes from v_1 to v_n.

Definition (Connected)

A graph G is *connected* if there is a walk between any two vertices v and w in G.

Definitions I won't use without explaining

- A *trail* is a walk that doesn't repeat any edges
- A *path* is a walk that doesn't repeat any vertices

Bipartite graphs

Definition (Bipartite graphs)

A graph G is *bipartite* if we can colour every vertex either blue or red so that every edge goes between a blue vertex and a red vertex.

Definition (Complete bipartite graphs)

The *complete bipartite graph* $K_{m,n}$ consists of $m + n$ vertices, m coloured red, n coloured blue, and an edge between any red vertex and and any blue vertex.

Examples

Another way to characterise bipartite graphs

Lemma

A graph G is bipartite if and only if it doesn't have any cycles of odd length (i.e., subgraphs of the form C_{2k+1}).

Bipartite $\implies$ no odd cycles:

Subgraphs of bipartite graphs are bipartite

No odd cycles $\implies$ Bipartite:

Try to colour G by distance from v

Definition (Distance)

Let G be connected, and let v, w be two vertices. The *distance from v to w* is the least number of edges in any walk from v to w.