Chromatic Polynomial

Rather than just knowing whether we *can* colour a graph G with k colours, we can *count* the different colourings that are possible.

Definition

Let G be a simple (why?) graph, and let $k \geq 1$ be an integer. The *chromatic polynomial* $P_G(k)$ is the number of different ways to colour the vertices of G with k colours, so that adjacent vertices have different colours.

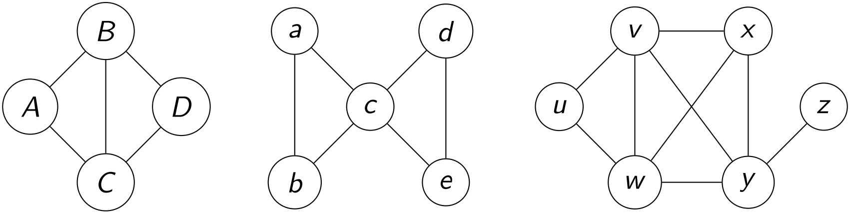
Remarks:

- It is *not* obvious from the definition that $P_G(k)$ is a polynomial. Prove next lecture.
- $P_G(k) = 0 \iff 0 \leq k < \chi(G)$

In easy examples, can colour vertex by vertex:

Colour vertex v_1, then vertices adjacent to v_1, then..

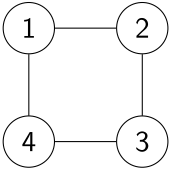
- $P_{E_n}(k) = k^n$
- $P_{K_n}(k) = k(k-1)(k-2)\cdots(k-n+1)$
- If T is a tree with n vertices, then $P_T(k) = k(k-1)^n$



G $\qquad$ H $\qquad$ Γ

What patterns do you notice?

A harder example: C_4



- k ways to colour vertex 1
- $k - 1$ ways to colour vertex 2
- $k - 1$ ways to colour vertex 3
- Don't know if v_1 and v_3 have same colour

Case 1: v_1 and v_3 have the same colour

- k choices for this colour
- Then $k - 1$ choices for each of v_2 and v_4

Case 2: v_1 and v_3 have the same colour

- k choices for v_1, then $k - 1$ choices for v_3
- $k - 2$ choices for each of v_2 and v_4

Combining the cases, we see:

$$P_{C_4} = k(k-1)^2 + k(k-1)(k-2)^2 = k(k-1)(k^2 - 3k + 3)$$

The two cases are themselves chromatic polynomials!

The type of reasoning we used to find the chromatic polynomial of C_4 will work to find the chromatic polynomial of any graph; however, many cases might need to be considered, and the argument will get quite complicated.

It will help to repackage the reasoning

Case 1: v_1 and v_3 are the same colour

If they're the same colour, then we can make them same vertex...

Case 2: v_1 and v_3 are different colours

If there's an edge between two vertices, then they need to be different colours.

Generalizing the observation we just made

A lemma, with some definitions baked in

Suppose that G is a graph, and x and y are two vertices that aren't adjacent. Define:

- G_{+xy} to be the graph with the edge xy added
- $G_{x=y}$ to be the graph with x and y identified

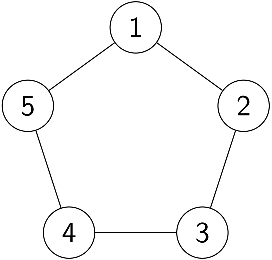
Then:

$$P_G(k) = P_{G_{+xy}}(k) + P_{G_{x=y}}(k)$$

Proof.

Consider the colourings of G. In some of them x and y will be colours, and in others x and y will be the same colour. The colourings in the first case are exactly the colourings of G_{+xy}, the colourings in the second are the colourings of $G_{x=y}$. □

The chromatic polynomial of C_5



Three cases, but two are the same:

- Case 1: 1,3 and 4 all have different colours
- Case 2: 1 and 3 have the same colour
- Case 3: 1 and 4 have the same colour

By symmetry, Cases 2 and 3 are the same.

- Case 1 gives: $k(k-1)(k-2)^3$
- Cases 2 and 3 each give: $k(k-1)^2(k-2)$

$$\begin{aligned} P_{C_5}(k) &= k(k-1)(k-2)^3 + 2k(k-1)^2(k-2) \\ &= k(k-1)(k-2)\left[k^2 - 4k + 4 + 2k - 2\right] \\ &= k(k-1)(k-2)(k^2 - 2k + 2) \end{aligned}$$

Deletion-Contraction: restructuring the lemma

The lemma as stated can be a bit awkward for induction:

- G_{+xy} has an extra edge; "bigger"
- $G_{x=y}$ has fewer vertices: "smaller"

Rewrite so $H \cong G_{+xy}$ is the "starting" graph.

Lemma (Deletion-Contraction)

Let H be a graph, and let e be an edge in H. Then

$$P_H(k) = P_{H\setminus e}(k) - P_{H/e}(k)$$

The edge e is between xy

$$H = G + xy, \qquad H \setminus e = G, \qquad H/e = G_{x=y}$$

Example

Re-do finding $P_{C_4}(k)$ and $P_{C_5}(k)$ using deletion-contraction.