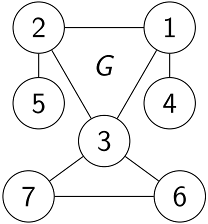
Where were we? Colouring graphs

- Chromatic number $\chi(G)$: colour vertices with fewest colours
- Chromatic index $\chi'(G)$: colour edges with fewest colours
- Chromatic polynomial $P_G(k)$: number of ways to colour vertices with k colours

Some graphs: colour vertex by vertex

$$P_G(k) = k(k-1)^4(k-2)^2$$



In general: need case-by-case

Examples we looked at: C_4, C_5.

Today: prove $P_G(k)$ is a polynomial

Message: Deletion-Contraction well suited for induction.

Chromatic polynomial is a polynomial

Theorem

Let G be a simple graph. Then $P_G(k)$ is a polynomial in k. Moreover, if G has n vertices and m edges, then

$$P_G(k) = k^n - mk^{n-1} + \textit{lower order terms}$$

Proof idea:

Induct on the number of edges using deletion-contraction.

Base case: $m = 0$

If G has no edges and n vertices, then $G = E_n$ empty graph.
$P_{E_n} = k^n$ is a polynomial of the right form.

Inductive step

Assume that G has $m > 0$ edges and n vertices, and that for any graph H with $\ell < m$ edges and p vertices, we have $P_H(k) = k^p - \ell k^{p-1} + \cdots$.

Let $e \in G$ be any edge:

- $G \setminus e$ has n vertices and $m - 1$ edges
- G/e has $n - 1$ vertices and *at most* $m - 1$ edges

So by the inductive hypothesis, theorem holds for $G \setminus e$ and G/e

So applying Deletion-Contraction:

$$\begin{aligned} P_G(k) &= P_{G \setminus e}(k) - P_{G/e}(k) \\ &= \left(k^n - (m-1)k^{n-1} + \cdots\right) - \left(k^{n-1} - \cdots\right) \\ &= k^n - mk^{n-1} + \cdots \end{aligned}$$

Which is what we needed to show. $\square$

Odds and Ends

Deletion-Contraction as an algorithm

- Can always find $P_G(x)$ by iterating deletion-contraction
- In practise, often faster to add edges

Information in $P_G(k)$

- $n =$ Number of vertices $=$ degree $P_G(k)$
- $m =$ Number of edges, $P_G(k) = x^n - mx^{n-1} + \cdots$
- $\chi(G)$ is the lowest k with $P_G(k) \neq 0$.

June Huh got the Field's Medal in 2022 for work on $P_G(k)$

- Roughly, proved coefficients of $P_G(k)$ are monotonic
- Used ideas from Algebraic Geometry

TellUsFeedback – I read and think about this!

MAS341 Graph Theory (SPRING 2023~24)

Students FO

Calculating the chromatic polynomial of C_n

Let e be any edge of C_n. Then:

- $C_n/e \cong C_{n-1}$
- $C_n \setminus e = P_n$, a tree, so $P_{P_n}(k) = k(k-1)^{n-1}$

So we should be able to find $P_{C_n}(k)$ using induction, but need to "guess" the formula first.

$$
\begin{aligned}
P_4(k) =& k^4 - 4k^3 + 6k^2 - 3k \\
P_5(k) =& k^5 - 5k^4 + 10k^3 - 10k^2 + 4k \\
P_6(k) =& k^6 - 6k^5 + 15k^4 - 20k^3 + 15k^3 - 5k \\
P_7(k) =& k^7 - 7k^6 + 21k^5 - 35k^4 + 35k^3 - 21k^2 + 6k
\end{aligned}
$$

Calculating the chromatic polynomial of C_n

Let e be any edge of C_n. Then:

- $C_n/e \cong C_{n-1}$
- $C_n \setminus e = P_n$, a tree, so $P_{P_n}(k) = k(k-1)^{n-1}$

So we should be able to find $P_{C_n}(k)$ using induction, but need to "guess" the formula first.

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P_7(k) =& k^7 - 7k^6 + 21k^5 - 35k^4 + 35k^3 - 21k^2 + 6k
\end{aligned}
$$

Looks like:

$$P_n(k) = (k-1)^n + (-1)^n(k-1)$$

Inductive proof that $P_{C_n}(k) = (k-1)^n + (-1)^n(k-1)$

Base case: $n = 3$

Plug in $n = 3$, get $k(k-1)(k-2) = P_{C_3}(k)$.

Inductive step:

Get to assume: $P_{C_{n-1}}(k) = (k-1)^{n-1} + (-1)^{n-1}(k-1)$

- $C_n \setminus e = P_n$, a tree, so $P_{C_{n-1} \setminus e} = k(k-1)^{n-1}$
- $C_n / e = C_{n-1}$, so $P_{C_n/e}(k) = (k-1)^{n-1} + (-1)^n(k-1)$.

Plugging into deletion-contraction:

$$\begin{aligned}
P_{C_n}(k) &= P_{C_n \setminus e}(k) - P_{C_n/e}(k) \\
&= k(k-1)^{n-1} - \left[(k-1)^{n-1} + (-1)^{n-1}(k-1)\right] \\
&= k(k-1)^{n-1} - (k-1)^{n-1} - (-1)^{n-1}(k-1) \\
&= (k-1)^{n-1}\left[k-1\right] + (-1)^n(k-1) \\
&= (k-1)^n + (-1)^n(k-1)
\end{aligned}$$