Gluing formulas: intro

Lemma

If G is the disjoint union of G_1 and G_2, then

$$P_G(k) = P_{G_1}(k)P_{G_2}(k)$$

Proof.

Colouring G is exactly the same as colouring G_1 and G_2 independently. □

Gluing formulas: when G isn't *quite* a disjoint union

Idea: Colour G_1, then extend to a colouring of G_2.

Gluing formulas: statements

Lemma

If G is made by gluing G_1 and G_2 along a vertex v, then:

$$P_G(k) = \frac{1}{k} P_{G_1}(k) P_{G_2}(k)$$

Proof.

First colour G_1 in any of the $P_{G_1}(k)$ ways. Now, vertex v of G_2 is already coloured, but none of the rest. Since the colours are interchangable, exactly $1/k$ of the ways of colouring G_2 will have the right colour at v. □

Lemma

If G is made by gluing G_1 and G_2 along an edge e, then

$$P_G(k) = \frac{1}{k(k-1)} P_{G_1}(k) P_{G_2}(k)$$

Find $P_G(k)$ for the following graphs