Second unit: Algorithms

Some topics in Decision Maths; mostly "easy" points on exam.

Largely about optimization problems:

- Kruskal and Prim's algorithms for cheapest spanning trees
- Dijkstra's algorithm: shortest path between two points
- Travelling Salesperson problem

In the real world: computers run these algorithm

From pure math perspective, interesting bits are:

- Proving the algorithm works as advertized
- Analyzing speed of algorithm – can you go faster?

First topic: Prüfer code

How many trees on n vertices are there?

Two interpretations of counting trees

Count isomorphism classes of trees. “unlabelled trees”

- This is what we do when we count isomers
- No nice answer

Count labelled trees on n vertices

- Vertices are no longer interchangeable
- $n!$ ways to label an unlabeled tree
- Symmetries mean some produce the same labelled tree

n	1	2	3	4	5	6	7
Unlabelled trees	1	1	1	2	3	6	11
Labelled trees	1	1	3	16	125	1296	16807

“Cayley’s” formula

Theorem (Borchardt 1860, Cayley 1889)

There are n^{n-2} labelled trees on n vertices.

Original proof used determinants.

Prüfer code: another way to prove Cayley’s formula

Bijection between *Trees* and *Codes*

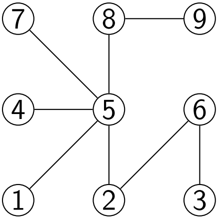
- $T_n = \{\text{labelled trees on } n \text{ vertices}\}$
- $C_n = \{(a_1, a_2, \ldots, a_{n-2}) : a_i \in \{1, 2, \ldots, n\}\}$
- Build a bijection between T_n and C_n
- $|C_n| = n^{n-2}$

Bijective Proofs

In combinatorics, bijective proofs often give more...

How to write down a labelled trees?

Record the edges:



Each column is an edge

1	5	2	6	5	7	8	8
5	2	6	3	4	5	9	5

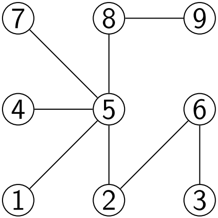
- Records $2n - 2$ numbers between 1 and n
- *Many* ways of recording same tree

Need to record the edges in a canonical way

"Canonical" means: without arbitrary choices

Prüfer code: iteratively remove lowest leaf

1. Find lowest leaf ℓ of T
2. Record edge e connecting it to rest of tree
3. Delete ℓ and e to get a simpler tree T'
4. Repeat process with T'



Parent	$\underline{5}$	$\underline{6}$	$\underline{5}$	$\underline{2}$	$\underline{5}$	$\underline{5}$	$\underline{8}$	9
Leaf	1	3	4	6	2	7	5	8

The underlined numbers form the Prüfer code

Non-underlined numbers are a permutation of 1-9. Why?

To see Prüfer code is a bijection, construct inverse

Given a Prüfer code, how to fill in empty boxes?

Parent	5	6	5	2	5	5	8	
Leaf								

Numbers in the Prüfer code were parents

- So the numbers in Prüfer code can't be leaves
- We deleted lowest leaf first
- Thus, first leaf is lowest number not in the Prüfer code

To reconstruct tree from code:

- Find lowest number not used yet that's not in remaining code
- Delete the first column
- Iterate; last two numbers left are the last edge

Cayley's enrichment: keep track of degrees of vertices

Parent	$\underline{5}$	$\underline{6}$	$\underline{5}$	$\underline{2}$	$\underline{5}$	$\underline{5}$	$\underline{8}$	9
Leaf	1	3	4	6	2	7	5	8

$$\begin{aligned}\text{Degree of vertex } i &= \text{number of times } i \text{ appears in table} \\ &= \text{number of times } i \text{ appears in Prüfer code} + 1\end{aligned}$$

Corollary

The number of labeled trees on n vertices where for each i, vertex i has degree d_i is:

$$\frac{(n-2)!}{\prod (d_i - 1)!}$$