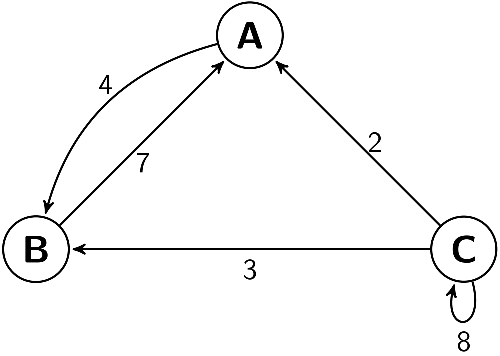
Today: Shortest and Longest paths

But first a basic definition.

Definition

A *directed graph* is one where each edge has a chosen starting and ending point, usually indicated with arrows.



Walks in directed graphs:

You can only travel the edge in the direction the arrow shows.

Dijkstra's algorithm for shortest path

Input:

A weighted (possibly directed) graph G and starting vertex $v \in G$

Output:

For every vertex $w \in G$ a list of all shortest paths from v to w

Initialize:

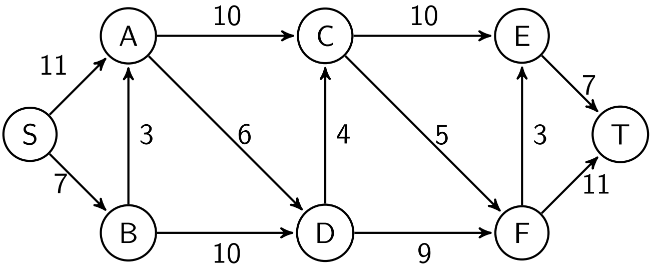
From starting vertex v list every edge out of v as a poentential shortest path to corresponding vertex w

Iterate:

- Choose w with cheapest potential shortest path and make these paths permanent
- Update list of potential paths by adding edges out of w to the shortest paths to w and checking if they're cheaper than known paths

Example graph from the 2008 Exam

Find all shortest paths from S to T.



Add on:

- Which edges if made a little longer would make the distance from S to T longer?
- Which edges if made a little shorter would make the distance from S to T shorter?

The main idea of why Dijkstra's works:

Suppose we're at the step were Dijkstra's decides the cheapest path to w.

Then:

A cheaper path to w would have to go through a vertex u we haven't found the cheapest path to yet.

But!

Even *getting* to u costs more than our cheapest path to w.

An observation:

Dijkstra's algorithm depends on the edge weights being non-negative!

Culture: performance of Dijkstra's algorithm

In a limited sense, Dijkstra's algorithm is optimal:

If *all we know* is that we have a weighted graph, then you can't do better than Dijkstra's algorithm.

In practise, often not very good:

When finding path from Sheffield to Edinburgh, Dijkstra's algorithm explores every street in London.

Real world maps have extra information:

It's easy to calculate the distance between two points as the crow flies, and we know the driving distance has to be at least that large.

The A* algorithm avoids searching London:

Supposes we have an easy to calculate "heuristic distance" $h(v, w)$, that is a lower bound for the actual distance $d(v, w)$.

Finding the longest path

Scheduling a large problem with many parts

For example, building a house.

- Some can be done at same time: (finishing interior rooms)
- Some need to be done in order: (foundation before walls)
- How early could whole project be finished?

Solution: longest path

- Encode tasks as edges in directed weighted graph
- Edge e follows edge f if task f requires task e
- Length of longest path is the shortest time to complete project

Building the directed graph from a list of tasks and dependencies can require a few tricks and won't be tested.

Longest path might not exist:

A necessary assumption:

If the graph has a directed cycle, we could get an infinitely long path by repeating graph over and over again.

Definition

A directed graph G is *acyclic* if it has no directed cycles.

Graphs in scheduling applications are acyclic:

Otherwise we'd have a cycle of tasks that all depend on each other and we could never start the project!

Ordering vertices / "topological sort"

If G is acyclic it's easy to order the vertices of G so that if there's an edge from v to w, then w comes after v.

The longest path algorithm:

Initializing:

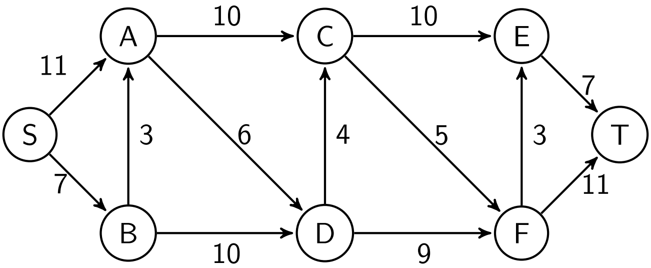
- Topologically sort the vertices of G
- List every edge of starting vertex as potential longest path

Iterate:

- Make the potential longest path to the first vertex w on the list permament
- Update the list of potential longest paths adding edges out of w to longest paths to w and seeing if they create new longest paths

Example graph from the 2008 Exam

Find all longest paths from S to T.



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- Which edges if made a little longer would make the longest path from S to T longer?
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