The Travelling Salesperson Problem

Informally:

A Travelling Salesperson wants to start from home, visit every city on their list, then return home for as cheaply as possible.

Definition

The *Travelling Salesperson Problem*, or TSP, is the following. Given a weighted graph G (usually G is the complete graph), what is the Hamiltonian cycle (i.e. closed walk visit every vertex) of cheapest weight?

The TSP is *Hard*

The TSP is at least as hard as finding Hamiltonian cycles

Let G be a graph n and vertex set V and edge set E.
Suppose we want to determine whether G has a Hamiltonian cycle.
Weight a complete graph on the vertex set V as follows: make every edge in G have weight 1, and every edge *not* in G have slightly higher weight:

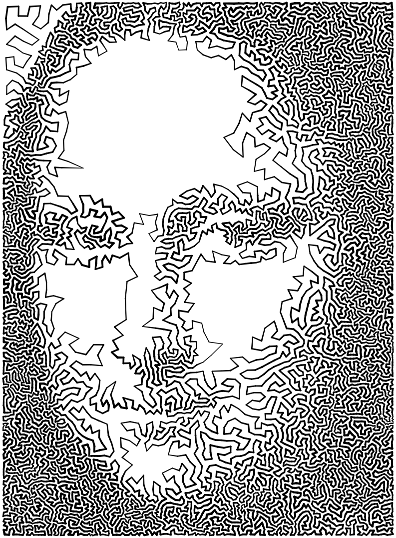
$$w(uv) = \begin{cases} 1 & uv \in E \\ 1+\varepsilon & uv \notin E \end{cases}$$

Then, G has a Hamiltonian cycle if and only if there is a solution to the TSP with weight n.

Bound the TSP instead of solving it

Can *almost* solve TSP in practice

Programs such as *Concorde* use sophisticated ideas to get solutions to TSP within a few percentage points on large data sets.



From *TSP Art*
by
Craig S Kaplan
and
Robert Bosch

To get an upper bound on TSP, find the weight of any Hamiltonian cycle

One greedy algorithm: nearest neighbour

Keeping going to the closest city you haven't been to

Why might nearest neighbour be bad?

Better heuristics exist:

- Nearest insertion: grow loop by inserting nearest city
- Christofide's Algorithm finds a Hamiltonian cycle at most 3/2 as expensive as the cheapest

These are more involved than nearest neighbour and still don't solve problem

To get a lower bound on TSP, can't use a cycle

Any Hamiltonian cycle has length *greater* than solution to TSP.

To find lower bound to TSP

- Pick a vertex $v \in G$
- Find a minimum weight spanning tree T on $G \setminus v$
- Find the two cheapest edges e_1 and e_2 out of v
- $w(T) + w(e_1) + w(e_2)$ is a lower bound

Need to be able to prove this gives lower bound...

Prove that $w(T) + w(e) + w(f) \leq TSP$

Suppose that C was the Hamiltonian cycle of minimum weight. We split the C into two pieces: the two edges f_1 and f_2 adjacent to v, and the rest, which we'll call R.

Edges adjacent to v:

e_1 and e_2 were the two cheapest edges next to v, so:

$$w(f_1) + w(f_2) \geq w(e_1) + w(e_2)$$

The rest:

- C a cycle, so $R = C \setminus v$ is a path
- Paths are special cases of trees
- R visits every vertex of $G \setminus v$
- Hence R a spanning tree and $w(R) \geq w(T)$

Example from the 2006 Exam

The following table encodes distances between towns in km:

A						
23	B					
10	21	C				
30	39	21	D			
57	45	48	45	E		
68	63	59	47	24	F	
75	67	66	54	24	11	G

Find lower and upper bounds to the TSP.