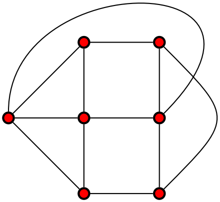
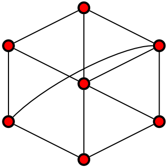
Which graphs are/aren't isomorphic? Prove it.

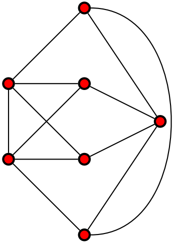
A



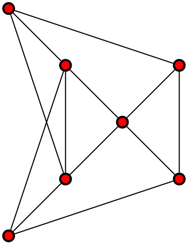
B



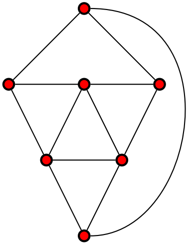
C



D

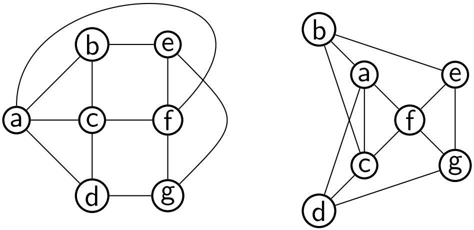


E



One solution to warm-up

- Graph B has a vertex of degree 5; others have degree sequence $[4, 4, 4, 3, 3, 3, 3]$, so none are isomorphic to B.
- In A, D, E, the three vertices of degree 4 all touch, but not in C, so none are isomorphic to C.
- In A, D, every vertex is adjacent to a vertex of degree 4, but not in E, so none are isomorphic to E.
- But we see below A is isomorphic to D:



A forest is a bunch of trees

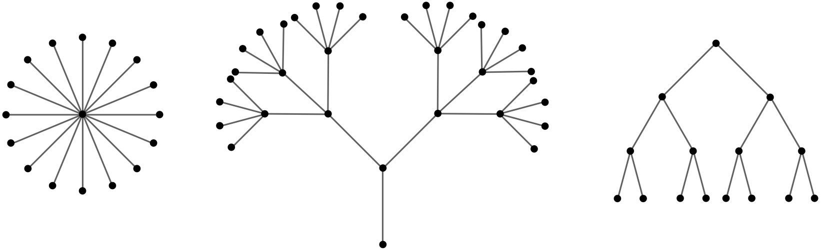


Figure: A forest of three trees

Definition

- A *forest* is a graph without cycles
- A *tree* is a connected graph without cycles

The Treachery of Definitions (After Magritte)

Figure: Ceci n'est pas un arbre (This is not a tree)

$\lfloor 13/2 \rfloor$ ways of looking a tree (After Wallace Stevens)

Proposition:

Let G be a graph with n vertices. The following are equivalent.

1. G is a tree (i.e., G connected but has no cycles)
2. There is a unique path in G between any two vertices
3. G is connected and has $n-1$ edges
4. G has no cycles and has $n-1$ edges
5. G is connected, but removing any edge disconnects G
6. G has no cycles, but adding any edge creates a cycle

Informally: Trees are Goldilocks graphs

- Trees have enough edges: they're connected
- Trees don't have too many edges: they have no cycles

Make like a tree and get out of here (After Biff Tannen)

Definition (Tree)

Let T be a tree. A vertex $v \in T$ is a *leaf* if it has degree 1.

Lemma

Let T be a tree with $2 \leq n < \infty$ vertices. Then T has at least two leaves.

Proof 1: See title of slide.

Pick an edge, and try to "leave" – that is, walk as far as you can.

- No loops, so you'll never return to where you are
- Finitely many vertices, so it can't go on forever

Eventually you'll get stuck – that's a leaf.

Pruning Trees

Part of Proposition:

If T is a tree with n vertices, then T has $n-1$ edges.

Proof: Induct on n

- Base case: $n = 1$
- Now assume that all trees with $n-1$ vertices have $n-2$ edges
- If T is a tree with n vertices, it has a leaf v (by Lemma)
- Delete v and the edge next to it to get a new tree T'
- T' has $n-1$ vertices, so $n-2$ edges, so T has $n-1$ edges.

Another use of the handshaking lemma

Part of Proposition:

If G is a connected graph with n vertices and $n-1$ edges, then G is a tree.

Proof: induct on n

- Base case: $n = 1$
- Assume proposition is true for all graphs with $n-1$ vertices
- Since G is connected, it has no vertices of degree 0
- Use handshaking to show G must have a vertex v of degree 1
- Delete v and the edge next to it to get a new graph G'
- G' is a tree, so G must have been as well

Chemistry

Chemical formulas encode degree sequences

Group→ ↓Period	1	2		3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1 H																		2 He
2	3 Li	4 Be												5 B	6 C	7 N	8 O	9 F	10 Ne
3	11 Na	12 Mg												13 Al	14 Si	15 P	16 S	17 Cl	18 Ar
4	19 K	20 Ca		21 Sc	22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 Cu	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
5	37 Rb	38 Sr		39 Y	40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
6	55 Cs	56 Ba	*	71 Lu	72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
7	87 Fr	88 Ra	**	103 Lr	104 Rf	105 Db	106 Sg	107 Bh	108 Hs	109 Mt	110 Ds	111 Rg	112 Cn	113 Uut	114 Fl	115 Uup	116 Lv	117 Uus	118 Uuo
			*	57 La	58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb		
			**	89 Ac	90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No		

Atom	C	N	O	H
Degree	4	3	2	1

Shortcuts around Carbon and Hydrogen

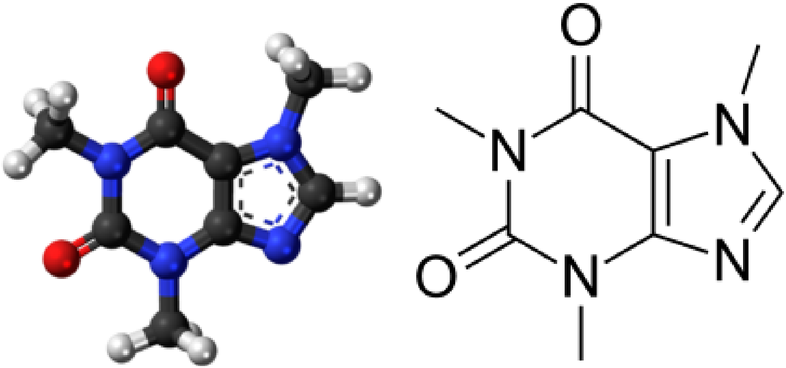


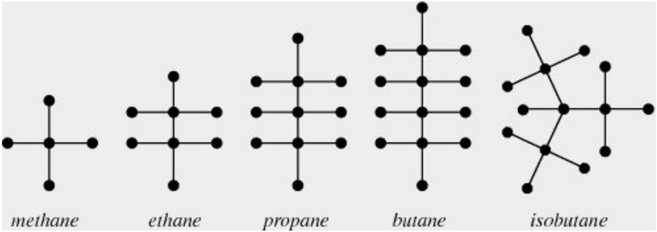
Figure: Two pictures of Caffeine

- Unlabelled vertices are Carbon
- Hydrogen not drawn; inferred to make degrees correct

Isomers are graphs with the same degree sequence

Definition

An *Alkane* is a molecule with formula $C_n H_{2n+2}$



Definition (Isomer)

Two different molecules are *isomers* if they have the same chemical formula.

Lemma: Any alkane is a tree.

Proof: Handshaking.

Question: How many isomers does $C_5 H_{12}$ have?