PROBLEM SET 3 SOLUTIONS

Part A (9 marks)

State Euler's Theorem for plane graphs, and use it to show that if a G is a plane graph with no vertices of degree less than 3 or cycles with length less than 4, then G must have at least 8 vertices and at least 12 edges. Give an example to show that these bounds cannot be improved.

Proof. Euler's Theorem for plane graphs states that if G is drawn on the sphere so that no edges cross, and the G has v vertices and e edges, and in the drawing there are f faces, then

$$v - e + f = 2$$

To prove the bound, we combine Euler's Theorem with handshaking theorems. Handshaking between vertices and edges, together with the fact that every vertex has degree at least 3 gives:

$$2e = \sum_v d(v) \geq 3v$$

and so $v \leq 2e/3$

Similarly, Handshaking between faces and edges, together with the fact that every cycle (and hence every face) has degree at least 4 gives

$$2e = \sum_f d(f) \geq 4f$$

and so $f \leq e/2$.

Substituting these inequalities into Euler's formula gives:

$$2 = v - e + f \leq 2e/3 - e + e/2 = e/6$$

and hence $e \geq 12$ as desired.

The inequality we currently have for v points the wrong way; instead we go back to Euler's formula, and first substitute our inequality for f, and then our inequality for e:

$$\begin{aligned} 2 &= v - e + f \\ &\leq v - e + e/2 = v - e/2 \\ &\leq v - 12/2 \end{aligned}$$

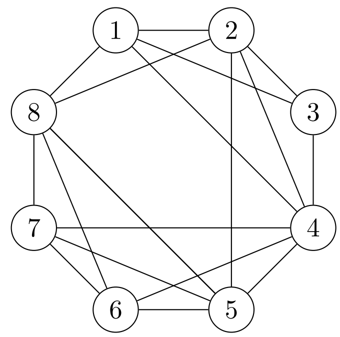
and so $v \geq 8$ as desired.

The cube graph is planar, has no cycles of length less than 4, and every vertex has degree 3, so it satisfies the hypotheses of our bound; furthermore, it has 8 verties and 12 edges, and so we see our bounds are tight. □

Part B (2 Marks)

Suppose that G is any graph with no vertices of degree less than 3 or cycles with length less than 4 – must G be planar? Justify your answer.

Proof. No, G need not be planar: $K_{3,3}$ is a counter example. Every vertex has degree 3, there are no loops, multiple edges, or triangles, so every cycle has length at least 4, and $K_{3,3}$ isn't planar. □

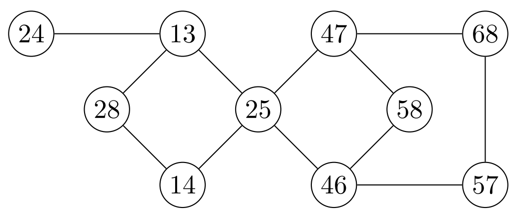


The rest of this question use the graph Γ show below.

Part C (5 Marks)

Use the Planarity Algorithm for Hamiltonian Graphs (with the Hamiltonian cycle 123456781) to show that Γ is not planar.

Proof. We find the crossing graph is as follows:



This crossing graph is not bipartite as it has the five cycle 25,47,68,57,46,25. Hence, Γ is not planar. □

Part D (5 Marks). State Kuratowski's Theorem, and use it to give another proof that Γ isn't planar.

Proof. Kuratowski's theorem states that a graph G is planar if and only if G does not have a subgraph that is a subdivision of K_5 or $K_{3,3}$.

The graph Γ shown has subdivisions of both K_5 and $K_{3,3}$.

For instance, to find a K_5, take as our vertices 4,5,6,7,8. Γ contains all edges between these vertices except the edge between 4 and 8, but we can connect these through 1, for instance.

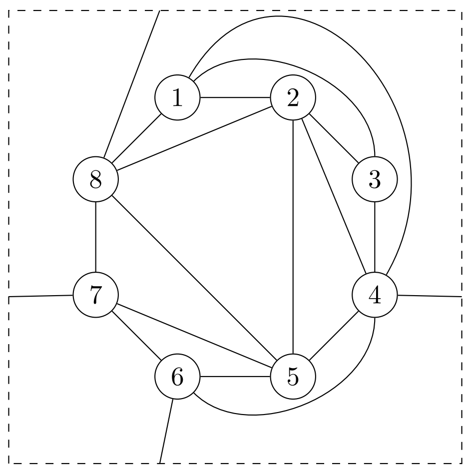
An example of a $K_{3,3}$ is to take 2,6,7 as our red vertices, and 4,5 and 8 as our blue vertices, then already every red vertex is adjacent to every blue vertex and we don't even need to subdivide the $K_{3,3}$.

Only one such subgraph was necessary, and there are undoubtedly others; after exhibiting such a subgraph, you should say, and "and so by Kuratowski's theorem Γ is not planar".

□

Part E (4 Marks). Draw Γ on the torus so that no edges cross.

Proof. The work done to find the crossing graph can help with this: we see that the graph was almost planar, with really only one 5 cycle as a problem. If we draw both 47 and 68 outside (which we can do without having them cross, thanks to the identifications of the edges of the square), then we've broken up that 5 cycle, and we can draw 24, 28, 25, 58, and 57 inside the octagon with no crossings, and draw the remaining edges outside, for example, like this:



Other drawings are possible, of course. □