MAS439 problem guidelines

I have provided you with a substantial collection of practice problems to work alongside the lectures; I hope that all of you will follow my suggestions, and that you will consult this list of problems regularly to both extend and enrich your understanding of what happens during the lectures. I will also ask you to hand in a problem or three every week as this will form the entirety of your assessment for this module: *do not take these problems lightly.* I have written this document to guide you in your quest to achieve mastery over the material we will cover in commutative algebra, and to give everyone involved a clear idea of what we can expect from each other. The assessed problems will be very similar in nature to the practice problems, and doing more of them will only improve your commutative algebra and proof-writing skills.

1. What does a solution look like?

While most of what I write below might seem silly or completely obvious, I think it is all very important stuff. Whether you decide to continue on and become a professional mathematician or not, perhaps the most important skills you should take from this degree are the ability to think logically and the ability to convey those logical thoughts to another person. You get practice doing both of those activities when you write up a proof for another person to read; you may never write another proof after this year, but I still maintain that every clear, logical, and readable proof you write is worth every minute you spend doing it, regardless of your future career. So, here are some tips when you are sitting down to write up a solution to a problem you have solved.

- I cannot stress this enough: write in complete sentences! The difference between writing in complete sentences and trying to get the same ideas across using just symbols and a few phrases can literally be the difference between getting half credit and getting full credit on a question.
- To go along with your complete sentences, I invite you to use proper punctuation. While American punctuation differs from British punctuation noticeably, both still perform the same function: they provide structure, and help the reader organize the words on the page. While I won't claim that punctuation makes as big of a difference in what mark you get as writing in complete sentences can, being strict on yourself in this area will make you focus on having clearly written work, and that does make a substantial difference.
- I suggest using LaTeXto write up your final solution. It might seem slow at first, but you will get faster, and it is good practice for your projects later in the year. More importantly, you will dramatically improve your solutions if you take the following easy steps.
  (1) Work through the problem.
  (2) Type it up using LaTeX.
  (3) Wait a couple hours. (Yes, this is an important step.)
  (4) Print out and read what you wrote, trying to have as open a mind about it as possible.
  (5) Mark your own solution, and then fix everything you found wrong with it.

The payoff is in the last step, but having a *typed* version of your solution will mean that you see errors more easily, as well as providing an easy way to produce another, better version of your solution - just print off the corrected copy!

- Make sure your solution has a clear structure. Explain what you are trying to prove, why those things address the question, and maybe some indication of how you are going to prove them (by contradiction, induction, etc.). Then follow through with your strategy, clearly marking each step. If you need some notation, make it obvious that you are defining some; if you are using a result or proof from the lectures, say that too. Generally speaking, structure your proof so that someone you have never met before could read it without any difficulty.
- Often students have trouble situating calculations inside of a proof in a way that is seamless and natural. One way to address this problem is to explain clearly what you are calculating, then give the calculation separately; if you are using LaTeX, using something like an array or equation environment is recommended. Another thing to remember is that an equal sign is a verb! Read this out loud: $5 = 4 + 1$. (Bonus fun to be had by labelling the parts of speech of each number.) That mathematical equation is a complete sentence, and should be treated like one in any solution.
- Finally, make sure you have a conclusion. Standard advice for writing any proof is say what you are going to prove, prove it, then say what you just proved. Sounds like overkill, but it can make your solution much easier for someone else to read. And the point of all this is not just to prove new things, but also to be able to prove them to *other people.*

2. What kind of feedback will I get?

You have done a problem, typed up a beautiful solution, and handed it in. That was your part of the deal, now comes mine. I will read your work, and give you feedback on it. That feedback will generally fall into a few categories.

- Correctness: Obviously I will tell you if your solution isn't correct, and I will try to say what you could do to fix it. In the unlikely occurrence that your solution is really headed in the wrong direction and needs to be entirely scrapped, I will have to inform you of that fact, and I will try to give you some hints.
- Style: Since *how* you write up your proof is just as important as the thoughts you were trying to convey, I will comment on that as well. These comments could be quite mundane ("This is not a complete sentence."), or they could point to more serious issues. Your solutions *will* be marked on how well you have written up your proofs, so don't dismiss this part of your assignment.
- Every few weeks, I will make sure to have time to talk to all of you about your progress in MAS439. I will tell you, generally, how you are doing in the module, and what you should be thinking about in the future to improve. I will probably pull out some assignments you have done and point to specific places where you could have done a better job. Remember that my goal here is to both help you learn more mathematics and help you learn how to better explain that mathematics to others.

3. Sample solutions with sample feedback

Here is a sample of what a homework problem might look like (this one is super easy on purpose), and two possible solutions. Feedback for the solutions is indicated in italics; some of this feedback is what you might see had you handed this in, some is just there as a reminder.

Problem. Show that the function $f : \mathbb{Z} \to \mathbb{Z}/2$ which sends an integer a to a mod 2 is a ring homomorphism.

Solution # 1: First, we will fix some notation. The ring $\mathbb{Z}/2$ has two elements, $0, 1$, and both addition and multiplication with these are performed as if they were the integers of the same name with the exception that $1 + 1 = 0$. In particular, 0 is the identity element for addition, and 1 is the identity element for multiplication. Then the function f sends a to 0 if a is even and sends a to 1 if a is odd. *Establishing notation for everything involved is always good practice.*

To show that f is a ring homomorphism, we must show that it preserves identities (both kinds), and that it preserves addition and multiplication. *Saying what you have to prove to complete the problem is good, and should start a paragraph.* We will do the identities elements first, then turn to addition and multiplication. *Establishing a clear order your proof will take helps the reader.*

Since 0 (the integer) is even, $f(0) = 0$, and therefore f preserves the identity element for addition. Similarly, the integer 1 is odd, so $f(1) = 1$ and f preserves the identity for multiplication. *Note here that this part of the proof is much clearer since I have established notation for the elements of* $\mathbb{Z}/2$.

Let a, b be a pair of integers. To show that f preserves addition, we must show that $f(a + b) = f(a) + f(b)$. Now $a + b$ is even when either a, b are both even or both odd. If they are both even, then

$$\begin{aligned} f(a) + f(b) &= 0 + 0 \\ &= 0 \\ &= f(a + b); \end{aligned}$$

if they are both odd, then

$$\begin{aligned} f(a) + f(b) &= 1 + 1 \\ &= 0 \\ &= f(a + b). \end{aligned}$$

This shows that when $a+b$ is even, $f(a+b) = f(a)+f(b)$. *Note that we summarize precisely what has been shown in this calculation.* To complete the proof that f preserves addition, we must also check this equation when $a+b$ is odd. In this case, either a is even and b is odd, or a is odd and b is even. Without loss of generality, we can assume the former. *There is no reason to do the same calculation twice, but you need to say that is what is happening.* Thus we have the following calculation.

$$\begin{aligned} f(a) + f(b) &= 0 + 1 \\ &= 1 \\ &= f(a + b). \end{aligned}$$

Therefore f preserves addition when $a + b$ is odd as well. We have now checked that f preserves addition in both cases (either $a + b$ is even or odd), proving that it preserves addition. *Once again, the strategy is clearly summarized.*

Finally, we must prove that f preserves multiplication. We use a similar strategy, breaking this down into the cases when ab is even or odd. *This clearly explains the structure for this part of the proof. Generally speaking, every paragraph needs to start with an explanation of what is to come.* For ab to be odd, we must have that both a, b are odd. In this case,

$$\begin{aligned} f(a)f(b) &= 1 \cdot 1 \\ &= 1 \\ &= f(ab). \end{aligned}$$

If either a or b is even, then so is ab; assume that a is even (the case in which b is even is analogous). Then we have the following calculation.

$$\begin{aligned} f(a)f(b) &= 0 \cdot f(b) \\ &= 0 \\ &= f(ab). \end{aligned}$$

In each case, we see that $f(ab) = f(a)f(b)$, so f preserves multiplication. *Once again, summarize this part of the proof.* We have now shown that f preserves both kinds of identity elements, addition, and multiplication, so f is a ring homomorphism. QED *Finally, a concluding statement for the whole proof.*

This solution is obviously a first-class solution, and would be treated as such if it were to appear as part of a piece of assessed coursework.

Solution # 2: It is obvious that $f(0) = 0, f(1) = 1$, so we check addition and multiplication. *This is true, but a bad place to start. You should always spell out any notation you need and your strategy first.* For addition, we must show that $f(a + b) = f(a) + f(b)$; assume that $a + b$ is even. *This is a good start!*

$$\begin{aligned} f(a + b) &\stackrel{?}{=} f(a) + f(b) \\ 0 &\stackrel{?}{=} f(a) + f(b) \\ 0 &\stackrel{?}{=} 0 + 0 \text{ or } 1 + 1, \end{aligned}$$

and since $1+1 = 0$, this is true in both cases, so $f(a+b) = f(a)+f(b)$ is true when $a + b$ is even. *This is a classically bad presentation of the right calculations. Never assume what you want to prove, or even make it look like you once considered doing so, as in the first line. Also make your assumptions clear before doing the calculation so you can avoid things like the third line.* When $a + b$ is odd, then one of them is odd and the other is even. Then the calculation is pretty much the same, so done with addition. *Once again starts well, then falls apart. Since the first calculation wasn't very good, using the same argument again is also a bad idea. Additionally, the second sentence is a little too informal for a proof; either something is the same, analogous (which maybe this calculation is, if set up correctly), or worthy of its own proof, never "pretty much" the same.*

Now we prove multiplication. *This isn't actually a sentence, even if it does convey the idea you are trying to get across.*

$$\begin{aligned} f(ab) &\stackrel{?}{=} f(a)f(b) \\ 0 \text{ or } 1 &\stackrel{?}{=} (0 \text{ or } 1) \cdot (0 \text{ or } 1) \end{aligned}$$

The only way the RHS is 1 is if both are 1 is if both are odd so ab is odd so LHS is also 1. *This sentence is very poorly constructed. It should probably be broken up into a couple sentences, and you should explicitly reference the integers a, b in the calculation above since that is what it is discussing. The ideas are clearly right, but the presentation makes it more confusing than it needs to be.* Otherwise 0, so multiplication is true as well. QED *Once again, this kind of thing gets commonly written on solutions, but is not a sentence for a number of different reasons. It also could use some additional justification, and then you should always end your proofs with a concluding remark.*

This solution deserves something like a 2.2 to me. Most of the right stuff is there, but it is poorly written, and if the problem were any more complicated I would bet that this student would leave out or forget some important points. Given that this problem alone would not likely constitute an entire week's worth of assessed work, I would wager that this student might stumble more seriously on other parts of the assignment that were more difficult.