MAS439 Lecture 19
Fields of fractions, Function fields

October 26th

Going back to Primary school

Remember how we construct the rational numbers $\mathbb{Q}$ out of the integers $\mathbb{Z}$:

- A rational number is an equivalence class of $(a, b) \in \mathbb{Z} \times \mathbb{Z}$, with $b \neq 0$ (we write $\frac{a}{b}$)
- $\frac{a}{b} \sim \frac{c}{d}$ if $ad - bc = 0$
- $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$
- $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$

These operations make $\mathbb{Q}$ into a field. The inverse of a/b is b/a, and $\mathbb{Z}$ sits inside $\mathbb{Q}$ as a subring, via $n \mapsto n/1$.
But we can generalize this...

The main definition

Let R be an integral domain. The *field of fractions* of R, denoted $K(R)$, is the field where the elements consists of equivalence classes of the form a/b, with $a, b \in R, b \neq 0_R$ with

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One has to check that $K(R)$ actually is a ring, and a field, which is a bit laborious but not difficult or enlightening.

Examples

$K(\mathbb{Z}) = \mathbb{Q}$

This is the example we used to motivate the definition of $K(R)$.

Rational functions

Recall that a function of one variable is called *rational* if it is the ratio of polynomials, e.g.

$$\frac{x}{x^2-1}, \quad \frac{x^3+3x}{ix^2+x+.5}, \frac{x^2-x}{x-1}, \quad \frac{x}{1}$$

It is clear the set of rational functions form a field, sometimes denoted $\mathbb{Q}(t), \mathbb{R}(t), \mathbb{C}(t)$. Then $\mathbb{Q}(t) = K(\mathbb{Q}(t))$.

More examples - nonisomorphic rings with isomorphic fields of fraction

$K(k) \cong k$

If R is already a field, then $K(R) \cong R$. So, we have $K(\mathbb{Z}) \cong \mathbb{Q} \cong K(\mathbb{Q})$.

But many more...

But maybe we're in between somewhere; let $S = \mathbb{Z}[x]/(2x-1)$. Then in S, we see that x is acting as $1/2$, and in particular we see powers of two have inverses in S. Then $K(S) \cong \mathbb{Q}$.

Why integral domains?

In the definition of $K(R)$, we required that R be an integral domain. Why?

Lemma

Suppose that R is not an integral domain. Then $K(R)$ is the trial ring.

Proof.

Let $uv = 0$ in R with $u, v \neq 0$. Then in $K(R)$, we have

$$1 = \frac{1}{u} \cdot u \cdot v \cdot \frac{1}{v} = 0.$$

□

Even if R isn't an integral domain, we can still add inverses to *some* things...

Function Fields

Definition

Let $X \subset \mathbb{A}_k^n$ be an affine variety, so by definition $k[x]$ is an integral domain. The *function field* $k(X)$ of X (sometimes called the *coordinate field*), is the field of fractions of the coordinate ring of X, i.e.

$$k(X) := K(k[X])$$

Example

For $X = \mathbb{A}_k^n$, we have $k[\mathbb{A}_k^n] = k[x_1, \ldots, x_n]$, and so $k(\mathbb{A}_k^n) = k(x_1, \ldots, x_n)$, the ring of rational functions in n variables.

More examples: varieties with isomorphic function fields

Example

Let $X = V(xy - 1) \subset \mathbb{A}^2_{\mathbb{C}}$. Then we see that

$$\mathbb{C}[X] = \mathbb{C}[x, y]/(xy - 1) \cong \mathbb{C}[x, x^{-1}]$$

I.e., the coordinate ring of X is the ring of *Laurent polynomials*. Thus:

$$\mathbb{C}(X) \cong \mathbb{C}(t) \cong \mathbb{C}(\mathbb{A}^1_{\mathbb{C}})$$

More examples, continued

Example

As a less trivial example, let $Y = V(x^2 - y^3)$ be the cuspidal cubic.
We have seen that $\mathbb{C}[Y] \cong \mathbb{C}[t^2, t^3]$. But

$$K(\mathbb{C}[t^2, t^3]) \cong \mathbb{C}(t) \cong \mathbb{C}[\mathbb{A}^1_{\mathbb{C}}]$$

since t^3/t^2 behaves just like t.

Elements of function fields as functions

Remember that we viewed $k[X]$ as polynomial functions from $X \to k$.
Thus, if $f = g/h \in k(X)$ we may view f as a partially defined function from $f \to k$, where if $a \in X$, we set $f(a) = g(a)/h(a)$, which makes sense as long as $h(a) = 0$.
We write $f : X \dashrightarrow k$ to distinguish that the function may not be defined everywhere.

Switching representatives

In first and second year, we viewed $\frac{x^2-1}{x-1}$ and $x+1$ as different functions, as the first isn't defined at $x = 1$. But in $\mathbb{C}(x)$, these two functions are defined to be equal.

Domains of rational functions

Definition

Let $f \in k(X)$ be a rational function.

- We say f is *regular* at $a \in X$ if there exists a representative g/h of f with $h(a) \neq 0$.
- The *domain* of f, written $\mathrm{dom}(f)$, is the set of points $U \subset X$ where f is regular.

Thus, a rational function $f \dashrightarrow k$ gives an actual function $f : \mathrm{dom}(f) \to k$.

Example of finding the domain

Let $X = V(y^2 - x^3)$, and consider $f = y/x \in \mathbb{C}(X)$. It is clear that f is well defined everywhere except perhaps at the origin $(0,0)$. We now show $(0,0) \notin \mathrm{dom}(f)$.
Suppose otherwise; then f has a representative p/q, with $q(0,0)0$.
Hence, q has a nonzero constant term, say $q(0,0) = c$.
But since $y/x \sim p/q$, we have $yq(x,y) = xp(x,y)$. But the y term of the left hand side is $cy \neq 0$, while the right hand side has no y term, contradiction.

There may not be one best representative

Consider $X = V(xy - zw) \subset \mathbb{A}^4_{\mathbb{C}}$, sometimes called the *conifold*.
Note that in $\mathbb{C}(X)$ we have $z/x = y/w$.
Now z/x is well defined as long as $x \neq 0$, while y/w is well defined as long as $w \neq 0$.

Regular functions

Definition

A rational function $f \in k(X)$ is called *regular* if its domain of definition $\mathrm{dom}(f)$is all X.

Lemma

Let $k = \overline{k}$. Then $f \in k(X)$ is regular if and only if we have $f \in k[x] \subset k(X)$; that is, if f has a representative of the form $f = g/1$.

Proof.

Idea:

- Show that $I(f) = 0 \cup \{h : \exists g/h = f\}$ is an ideal in $k[x]$.
- Apply the Nullstellensatz to $I(f)$.

□