Commutative Algebra Lecture 2

Last time

- Course set-up: 5 homeworks
- What and Why of normal subgroups (quotients!)
- Homomorphisms preserve structure (will review)
- Commutative Rings – can add and multiply
- Most important example: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}/n\mathbb{Z}, R[x]$. Others?

Today

- Types of elements and rings
- Test clicker system
- Start homomorphisms (Section 4 of notes)

Looking into the future

- The last HW question depends on Section 4
- Next session we will finish Section 4, motivate 5 and 6
- After that, roughly one section / lecture

Before next week's lecture...

- Read Sections 2-4 of the notes
- Email me questions / comments you have about them

The first Homework is due October 18th!

Basic definitions: types of elements

Definition

We say $r \in R$ is a *unit* if there exists an element $s \in R$ with $rs = 1_R$

Definition

We say that $r \in R$ is a *zero divisor* if there exists $s \in R, s \neq 0_R$ with $rs = 0_R$

Definition

We say that $r \in R$ is *nilpotent* if there exists some $n \in \mathbb{N}$ with $r^n = 0_R$

Examples!

Basic definitions: Types of rings

Definition

We say R is *field* if every nonzero element is a unit.

By convention, the trivial ring is not a field.

Definition

We say R is an *integral domain* if it has no zero divisors.

Definition

We say that R is *reduced* if it has no nilpotent elements.

Examples!

Theorem

R a field $\implies$ R an integral domain $\implies$ R is reduced

Clicker session: ttpoll.eu

Review: (Homo)-morphisms preserve structure

Objects

Often in math we study things that are sets with some extra sturcture (Groups, rings, fields, vector spaces, metric spaces, topological spaces, measure space, . . .).

Maps or Morphisms

In these situations, usually there is a notion of *map* or *morphism* between these objects – these are functions that "preserve the extra structure"

- Group homomorphisms preserve addition, units, inverses
- Vector space morphisms (linear maps) preserve addition and multiplication by scalars

What do we mean by "preserve structure"?

More specifically, recall that a group G has:

- an identity e
- a multiplication map $G \times G \to G : (g, h) \mapsto g \cdot h$
- an inverse map $G \to G : g \mapsto g^{-1}$.

Definition

A group homomorphism $\varphi : G \to H$ is a map of sets so that

1. $\varphi(e_G) = e_H$
2. $\varphi(g^{-1}) = \varphi(g)^{-1}$
3. $\varphi(g_1 \cdot g_2) = \varphi(g_1) \cdot \varphi(g_2)$

Warning: For us, "preserve the structure" doesn't have to be this striaghtforward.

What is a ring homomorphism?

Ring Homomorphisms

Definition

A ring homomorphism $\varphi : R \to S$ is a function so that

1. $\varphi(0_R) = 0_S$
2. $\varphi(1_R) = 1_S$
3. $\varphi(-r) = -\varphi(r)$
4. $\varphi(r + s) = \varphi(r) + \varphi(s)$
5. $\varphi(rs) = \varphi(r)\varphi(s)$

This is slightly more involved than the definition in the notes, because some of these properties follow from others...

Which?

Examples of ring homomorphisms!