Section 11: Polynomial algebras

Going to do this section a bit fast, as there's not that much there.
It formally introduce $k[x_1, \ldots, x_n]$, and answers the question:

Why are polynomial rings and their quotients important?

First answer: $k[x]$ satisfies a universal property.
Because of universal property:

$$R \text{ is a finitely generated } k\text{-algebra}$$

$$\Longleftrightarrow$$

$$R \cong k[x_1, \ldots, x_n]/I$$

Universal Property of $k[x]$

Lemma

$k[x]$ and x satisfies the following universal property: for any k-algebra S and any element $s \in S$, there is a unique k-algebra homomorphism $\varphi_s : k[x] \to S$ such that $\varphi_s(x) = s$.

Proof.

Plug s in for x. □

Lemma

If R, r is any k-algebra satisfying the universal property of $k[x]$, then there is a unique isomorphism between R and $k[x]$ identifying r with x.

Similarly, k-algebra homomorphisms $k[x_1, \ldots, x_n]$ to R are the same thing as n-tuples of elements $r_1, \ldots, r_n \in R$.

Finitely generated = Quotient of Polynomial Algebra

Finitely Generated $\implies$ quotient

Suppose R is generated by $r_1, \ldots, r_n$.

- The homomorphism $\varphi : k[x_1, \ldots, x_n] \to R$ sending x_i to r_i is surjective.
- By first isomorphism theorem $R[x_1, \ldots, x_n]/\ker(\varphi) \cong R$.

Quotient $\implies$ finitely generated

In the other direction, if $R = k[x_1, \ldots, x_n]/I$ for some ideal I, then R is generated by $[x_1], \ldots, [x_n]$.

So it might *seem* like it's restrictive to study $k[x_1, \ldots, x_n]/I$, but we're really studying finitely generated k-algebras.

Can we have infinitely many relations?

I.e., does I need to be finitely generated?

Section 12: Noetherian rings

Definition

A ring R is *Noetherian* if it satisfies the *Ascending Chain Condition*, or A.C.C., namely, if every ascending chain of ideals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

eventually stabilizes, i.e., there exists some N with $I_N = I_{N+1} = I_{N+2} = \cdots$.

Examples

- Any field k
- $\mathbb{Z}$ or more generally any principle ideal domain
- R Noetherian $\implies$ R/I Noetherian

Why study Noetherian Rings?

Because of this lemma:

Lemma

A ring R is Noetherian if and only if every ideal is finitely generated.

Proof.

$\Longrightarrow$ Assume I not f.g., try to generate, get contradiction.

$\Longleftarrow$
- Take an ascending chain I_n
- $I = \cup I_n$ is an ideal, hence $I = (r_1, \ldots, r_k)$
- If $\{r_i\} \in I_n$, then $I_n = I$,

$\square$

Hilbert Basis Theorem

Theorem

If R is Noetherian, then so is $R[x]$.

Proof.

Main ideas: look at leading coefficients, induct on degree. □

Corollary

Let k be a field or principle ideal domain. Then every ideal $I \subset k[x_1, \ldots, x_n]$ is finitely generated.

Proof.

k-Noetherian $\implies k[x_1, \ldots, x_n]$ Noetherian
$\implies$ every ideal of $k[x_1, \ldots, x_n]$ is finitely generated □

Hence: finitely generated k-algebras are finitely presented.