MAS332 Complex Analysis

Dr F.M.Hart (Office K12)

1 Complex numbers

1.1 Revision and Notation

1. The complex plane $\mathbb{C} = \{z : z = x + iy \text{ with } x \text{ and } y \in \mathbb{R}\}$. Since a real x can be written as $x + i0$, it can be viewed as being complex also. Obviously $\mathbb{R} \subset \mathbb{C}$.

   We know that $\mathbb{R}$ is an ordered set, i.e., if $x, y \in \mathbb{R}$ then precisely one of the following relations is satisfied (i) $x < y$, (ii) $x = y$, (iii) $x > y$.
   The **set $\mathbb{C}$, however, is <u>NOT AN ORDERED SET</u>**, i.e., it is not possible to define the inequality $z_1 < z_2$ sensibly for $z_1,\ z_2 \in \mathbb{C}$ and so inequalities such as $1 + i > 0$, $z > 0$, and $-5 - 6i < 10 + 37i\ldots$ are **<u>MEANINGLESS</u>.**
   Inequalities can only be used with complex numbers if they are essentially inequalities between real numbers, e.g., we could write $|z_1| < |z_2|$, or $\operatorname{Re} z_1 < \operatorname{Re} z_2$ etc.

2. If you have one equation involving complex numbers, you can equate real and imaginary parts and get two equations involving real numbers.

3. (a) If $z = x + iy$, then $\overline{z} = x - iy$ is the *complex conjugate* of z. (In some books, this is denoted z^*.)
   (b) $|z| = \sqrt{(x^2 + y^2)}$ is the *modulus* of z.

   (c) $\operatorname{Re}(x + iy) = x, \qquad \operatorname{Im}(x + iy) = y \qquad$ (**<u>NOT</u>** $\quad iy$) .

   Note that $|z| = |z - 0|$ is the distance between z and the origin. Similarly, $|z - a|$ is the distance between z and a.

   Thus the inequality $|z| < |w|$ for $z,\ w \in \mathbb{C}$, means that the point z is closer to the origin than w. For example $|2 - 2i| < |-3 - i|$.

4. $|z|^2 = z \times \overline{z}$ (both being $x^2 + y^2$). This is useful as $|\ |$ is hard to manipulate. For example,
$$\frac{1}{z} = \frac{1}{z \times \overline{z}} \times \overline{z} = \frac{x - iy}{x^2 + y^2}.$$

N. B. $|z|^2 = z\overline{z} = x^2 + y^2$ and so it is <u>NOT</u> the same as $z^2 = (x+iy)^2 = x^2 + 2ixy - y^2$ unless z is real.

5. If $z \neq 0$, then the line joining 0 to z makes an angle θ with the positive real axis $\mathbb{R}^+$. θ is called a value of $\arg z$, as is $\theta + 2\pi$, $\theta + 4\pi, \ldots$, and $\theta - 2\pi$, $\theta - 4\pi, \ldots$. So $\arg z$ has infinitely many values and is NOT a function.

6. The polar form is $z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta)$. Here $r = |z| \geq 0$ and θ is any value of $\arg z$. This is OK since (using $e^{i\alpha} = \cos\alpha + i\sin\alpha$),

$$e^{i(\theta+2n\pi)} = \cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi) = \cos\theta + i\sin\theta = e^{i\theta}$$

and so the ambiguity in θ does not give different values for $e^{i\theta}$. The form $z = re^{i\theta}$ is also often called modulus-argument form. I shall use either the term "polar form" or "modulus-argument form". You should be used to both names.

We recall that if

$$z \;=\; r(\cos\theta \;+\; i\sin\theta), \quad w \;=\; s(\cos\phi \;+\; i\sin\phi)$$

then

$$zw \;=\; rs(\cos(\theta+\phi) + i\sin(\theta+\phi)) \;\text{ and, if } s \neq 0, \quad \frac{z}{w} \;=\; \frac{r}{s}\left(\cos(\theta-\phi)+i\sin(\theta-\phi)\right).$$

So modulus-argument form is particularly useful when multiplying and dividing complex numbers, finding powers and roots of complex numbers.

7. **De Moivre's Theorem**: Let $\theta \in \mathbb{R}$ and $n \in \mathbb{N}$. Then

$$(\cos\theta \;+\; i\sin\theta)^n \;=\; (\cos n\theta \;+\; i\sin n\theta)$$

and $(\cos\theta \;+\; i\sin\theta)^{\frac{1}{n}}$ has precisely n different values, given by

$$\cos\left(\tfrac{\theta}{n} + \tfrac{2k\pi}{n}\right) \;+\; i\sin\left(\tfrac{\theta}{n} + \tfrac{2k\pi}{n}\right) \qquad (k = 0, 1, 2, \cdots (n-1)).$$

In terms of the $e^{i\theta}$ notation, we have $(e^{i\theta})^{\frac{1}{n}}$ has precisely n different values, given by

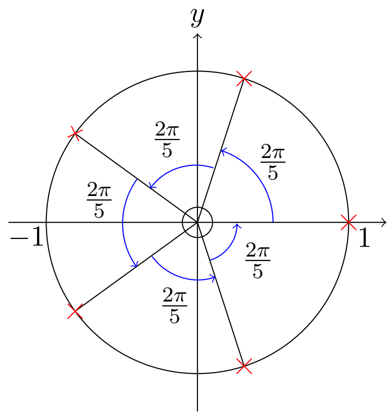
$$e^{i\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)} \qquad (k = 0, 1, 2, \cdots (n-1))\,. \qquad \square$$

x

For example, the five values of $1^{\frac{1}{5}}$ are

$$1\,, \quad \cos(\tfrac{2\pi}{5})+i\sin(\tfrac{2\pi}{5})\,, \quad \cos(\tfrac{4\pi}{5})+i\sin(\tfrac{4\pi}{5})\,, \quad \cos(\tfrac{6\pi}{5})+i\sin(\tfrac{6\pi}{5})\,, \quad \cos(\tfrac{8\pi}{5})+i\sin(\tfrac{8\pi}{5})\,.$$

These five values of the fifth root of unity are represented in an Argand diagram by five points evenly spaced round the unit circle.



If the nth roots of a complex number $r(\cos\theta + i\sin\theta)$ $\quad (r > 0)$ are required, then these are

$$\sqrt[n]{r}\,(\cos(\tfrac{\theta}{n} + \tfrac{2k\pi}{n}) \; + \; i\sin(\tfrac{\theta}{n} + \tfrac{2k\pi}{n})) \qquad (k = 0, 1, 2, \cdots (n-1))\,,$$

where $\sqrt[n]{r}$ is the positive real number a such that $a^n = r$.

For example, the roots of the equation $z^3 + 8 \;=\; 0$ are the cube roots of $\; -8$. They are

$$2(\cos(\tfrac{\pi}{3}) + i\sin(\tfrac{\pi}{3}))\ , \qquad -2\ , \qquad 2(\cos(\tfrac{5\pi}{3}) + i\sin(\tfrac{5\pi}{3}))\ ,$$

since $-8 \;=\; 8(\cos\pi + i\sin\pi)$.

8. Recall that

$$\begin{array}{rclrclrcl} e^{2\pi i} &=& 1, & e^{i\pi} &=& e^{-i\pi} = -1, & e^{\pi i/2} &=& i, \\ e^{-\pi i/2} &=& -i, & e^{2\pi i/3} &=& -1/2 + i\sqrt{3}/2, & e^{-2\pi i/3} &=& -1/2 - i\sqrt{3}/2. \end{array}$$

1.2 Examples

1. Express $\dfrac{\sqrt{3}+i}{1-i}$ (a) in the form $x + iy$ and (b) in modulus-argument form. Hence find the values of $\cos(5\pi/12)$ and $\sin(5\pi/12)$.

2. Find $(\sqrt{3}+i)^{48}$ and find the values of $(\sqrt{3}+i)^{1/48}$.

3. Evaluate $\left|\dfrac{a-b}{1-\overline{a}b}\right|$, when a, $b \in \mathbb{C}$, $a \neq b$, and $|a| = 1$.

4. Find the fourth roots of -1 and hence express $z^4 + 1$ as a product of two real quadratics.

Solutions

1. Multiply the numerator and the denominator by the complex conjugate of the denominator to obtain,

(a)
$$\frac{\sqrt{3}+i}{1-i} = \frac{\sqrt{3}+i}{1-i} \times \frac{1+i}{1+i} = \left(\frac{\sqrt{3}-1}{2}\right) + i\left(\frac{\sqrt{3}+1}{2}\right). \quad (1)$$

(b)
$$\frac{\sqrt{3}+i}{1-i} = \frac{2e^{\frac{\pi i}{6}}}{\sqrt{2}e^{-\frac{\pi i}{4}}} = \sqrt{2}e^{\frac{5\pi i}{12}} = \sqrt{2}\cos\left(\frac{5\pi}{12}\right) + i\sqrt{2}\sin\left(\frac{5\pi}{12}\right). \quad (2)$$

Equating real and imaginary parts,

$$\cos\left(\frac{5\pi}{12}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}, \qquad \sin\left(\frac{5\pi}{12}\right) = \frac{\sqrt{3}+1}{2\sqrt{2}},$$

using (1) and (2).

2. Use modulus-argument form $re^{i\theta}$ if you have to take powers. Thus

$$\left(\sqrt{3}+i\right)^{48} = \left(2e^{\frac{\pi i}{6}}\right)^{48} = 2^{48}e^{8\pi i} = 2^{48}.$$

Also, one value of $(\sqrt{3}+i)^{\frac{1}{48}}$ is $2^{\frac{1}{48}} \times e^{\frac{\pi i}{288}}$. Write $\alpha = 2^{\frac{1}{48}} \times e^{\frac{\pi i}{288}}$, say and let $\omega = e^{\frac{2\pi i}{48}}$. Then α, $\alpha\omega$, $\alpha\omega^2$,…,$\alpha\omega^{47}$ are the 48 values of the 48th roots of $\sqrt{3}+i$.

3. We use $a\overline{a} = 1$ to obtain

$$\left|\frac{a-b}{1-\overline{a}b}\right| = \left|\frac{a-b}{1-\frac{b}{a}}\right| = |a|\left|\frac{a-b}{a-b}\right| = 1.$$

<u>Note</u> In questions like this avoid using real and imaginary parts if possible.

4. Since $-1 = e^{\pi i}$ the fourth roots of -1 are $e^{\pi i/4}$, $e^{-\pi i/4}$, $e^{3\pi i/4}$ and $e^{-3\pi i/4}$. Hence

$$\begin{aligned} x^4+1 &= (x-e^{\pi i/4})(x-e^{-\pi i/4})(x-e^{3\pi i/4})(x-e^{-3\pi i/4}) \\ &= [x^2-(e^{\pi i/4}+e^{-\pi i/4})x+1][x^2-(e^{3\pi i/4}+e^{-3\pi i/4})x+1] \\ &= [x^2-2x\cos\frac{\pi}{4}+1][x^2-2x\cos\frac{3\pi}{4}+1] \\ &= [x^2-\sqrt{2}x+1][x^2+\sqrt{2}x+1] \end{aligned}$$

using

$$\begin{aligned} (e^{i\theta}+e^{-i\theta}) &= (\cos\theta + i\sin\theta) + (\cos(-\theta) + i\sin(-\theta)) \\ &= (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta) = 2\cos\theta. \end{aligned}$$

1.3 Inequalities

First note that

$$|\operatorname{Re} z| \le |z| \le |\operatorname{Re} z| + |\operatorname{Im} z| \quad \text{and} \quad |\operatorname{Im} z| \le |z| \le |\operatorname{Re} z| + |\operatorname{Im} z|$$

since $|x| \le (x^2+y^2)^{\frac{1}{2}} \le (|x|^2 + 2|x||y| + |y|^2)^{\frac{1}{2}} = |x| + |y|$ and a similar inequality holds for $|y|$. There is a clear geometrical interpretation of these inequalities: the length of one of the sides of a right-angled triangle is less than the length of the hypotenuse and the length of the hypotenuse is less than the sum of the lengths of the two shorter sides.

Theorem 1.1 *The triangle inequalities state that if z and $w \in \mathbb{C}$, then*

$$||z| - |w|| \le |z+w| \le |z| + |w|,$$

$$||z| - |w|| \le |z-w| \le |z| + |w|.$$

Proof. Consider

$$\begin{aligned}
|z+w|^2 &= (z+w)\overline{(z+w)} \\
&= (z+w)(\overline{z}+\overline{w}) \\
&= z\overline{z} + (z\overline{w} + \overline{z}w) \ + \ w\overline{w} \\
&= z\overline{z} + 2\operatorname{Re}(z\overline{w}) \ + \ w\overline{w} \\
&\le |z|^2 + 2|z\overline{w}| + w\overline{w} \\
&= |z|^2 + 2|z||w| + |w|^2 \\
&= (|z| + |w|)^2
\end{aligned}$$

Take positive square roots to obtain

$$|z+w| \le |z| + |w| \,. \tag{1}$$

If we replace w by $-w$ in equation (1), we obtain

$$|z-w| \le |z| + |-w| = |z| + |w| \,. \tag{2}$$

From equation (1), we have $|z| = |(z-w)+w| \le |z-w| + |w|$, giving

$$|z-w| \ge |z| - |w| \,. \tag{3}$$

Interchanging z and w in equation (3) gives

$$|z-w| = |w-z| \ge |w| - |z| \tag{4}$$

Equations (3) and (4) together give

$$|z-w| \geq \big||z|-|w|\big|\,. \tag{5}$$

Finally, replacing w by $-w$ in equation (5) to obtain

$$|z+w| \geq \big||z|-|w|\big|\,. \tag{6}$$

$\square$

Note The inequalities for $|z+w|$ and $|z-w|$ have the same form. Your choice of which part of the triangle inequalities to use in any application is governed by the form of the inequality you need to prove rather than whether you are looking at $|z+w|$ or $|z-w|$. The next set of examples illustrate this. **Be very wary when dealing with inequalities involving moduli of fractions and, in particular, be careful how you treat the denominator.**

1.4 Examples

1. Show that
$$\frac{1}{3} \leq \left|\frac{2z^2-1}{z+2}\right| \leq 3$$
for all z on $|z|=1$.

2. Show that $2 \leq |\,3z+4i| \leq 8$ for all z with $|\,z+1| \leq 1$.

Solutions

1. Using the triangle inequalities, we see that
$$|\;2z^2-1| \;\leq\; 2|z^2|+1 \;=\; 2+1 \;=\; 3$$
and
$$|\;2z^2-1| \;\geq\; \big||2z^2|-1\,\big| \;=\; 2-1 \;=\; 1\,,$$
for all z such that $|z|=1$. Thus
$$1 \;\leq\; |\,2z^2-1| \;\leq\; 3 \tag{1}$$
for all z such that $|z|=1$. Similarly, for $|z|=1$,
$$1 = 2-1 = \big||z|-2\big| \leq |z+2| \leq |z|+2 = 1+2 = 3.$$
Thus, for all z with $|z|=1$,
$$\frac{1}{3} \;\leq\; \left|\frac{1}{z+2}\right| \;=\; \frac{1}{|z+2|} \;\leq\; 1\,. \tag{2}$$
It follows from (1) and (2) that
$$\frac{1}{3} \;\leq\; \left|\frac{2z^2-1}{z+2}\right| \;\leq 3\,,$$
for all complex numbers z such that $|z|=1$.

2. First write $3z + 4i = 3(z + 1) + (4i - 3)$. (i.e. **Write the given expression in terms of something that you know something about.**) Then, using the triangle inequalities,

$$|3z + 4i| = |3(z + 1) + (4i - 3)| \leq |3(z + 1)| + |4i - 3| = 3|z + 1| + 5 \leq 3 + 5 = 8$$

$$|3z + 4i| = |3(z + 1) + (4i - 3)| \geq |\,|3(z + 1)| - |4i - 3|\,| = 5 - 3|z + 1| \geq 5 - 3 = 2\,,$$

giving

$$2 \ \leq\ |3z + 4i| \ \leq\ 8.$$

2 Special functions

2.1 The exponential

Possible definitions of e^z (or $\exp(z)$):

1. $e^z = \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$. The power series has infinite radius of convergence and so is defined everywhere on $\mathbb{C}$.
2. $e^z = \displaystyle\lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n$, again extending real result $\left(1 + \frac{x}{n}\right)^n \to e^x$ as $n \to \infty$.
3. $e^z = e^x(\cos y + i \sin y)$ if $z = x + iy$.

We will use (3) as our initial definition. After the section on power series we will, however, assume (1) (without proof) when we feel like it.

Definition 2.1 *Let $z = x + iy$. Then the exponential function is defined by*

$$e^z = e^x(\cos y + i \sin y). \qquad \square$$

Sometimes I write $\exp z$ in place of e^z. This has the advantage of emphasizing that the exponential function is a function rather than a power. In some circumstances it is also easier to read e.g. $\exp z^2$ is clearer than e^{z^2}.

Theorem 2.2

1. *If $z = x + iy$, then $|\,e^z| = e^x = e^{\operatorname{Re} z}$, and y is a value of $\arg(e^z)$. The numbers $y + 2n\pi \ \ (n \in \mathbb{Z})$ give all the values of $\arg(e^z)$,*
2. *For all $z, w \in \mathbb{C}$, $e^{z+w} = e^z \times e^w$,*
3. *The exponential function e^z is periodic with period $2\pi i$,*

4. For all $z \in \mathbb{C}$, $e^z \neq 0$ and $\frac{1}{e^z} = e^{-z}$.

Proof.

1. Let $z = x + iy$, then from the definition,

$$e^z = e^x(\cos y + i \sin y)$$

and this is modulus-argument form for e^z, as $e^x > 0$ for all real x. Thus

$$|\, e^z| = e^x = e^{\operatorname{Re} z} \quad \text{and one value of the argument of } e^z \text{ is } y\,.$$

2. For all $z, w \in \mathbb{C}$,

$$\begin{aligned} e^z e^w &= e^x(\cos y + i \sin y) \times e^u(\cos v + i \sin v) \\ &= e^{x+u}[(\cos y \cos v - \sin y \sin v) + i(\cos y \sin v + \sin y \cos v)] \\ &= e^{x+u}(\cos(y+v) + i \sin(y+v)) \\ &= e^{z+w}\,. \end{aligned}$$

3. For all $z \in \mathbb{C}$,

$$\begin{aligned} e^{z+2\pi i} &= e^z \times e^{2\pi i} \\ &= e^z(\cos(2\pi) + i \sin(2\pi)) \\ &= e^z\,. \end{aligned}$$

So the exponential function has period $2\pi i$ and takes the same value at all points $z + 2n\pi i$ $(n \in \mathbb{Z})$.

4. We know that $e^0 = 1$, so

$$e^z \times e^{-z} = e^{z-z} = e^0 = 1\,. \tag{1}$$

Now e^z and e^{-z} are two complex numbers whose product is 1 and so neither of them is zero. Thus $e^z \neq 0$ for all $z \in \mathbb{C}$. Hence, using (1),

$$e^{-z} = \frac{1}{e^z}\,.$$

$\square$

2.2 More functions

We define:

$$\begin{aligned} \cosh z &= \tfrac{1}{2}(e^z + e^{-z}), & \sinh z &= \tfrac{1}{2}(e^z - e^{-z}), \\ \cos z &= \tfrac{1}{2}(e^{iz} + e^{-iz}), & \sin z &= \tfrac{1}{2i}(e^{iz} - e^{-iz}). \end{aligned}$$

These functions are all defined on $\mathbb{C}$, and the formulae generalise our real knowledge. It is immediate that $\cos(-z) = \cos z$ (so cos is an even function) and $\sin(-z) = -\sin z$ (so sin is an odd function). Similarly, cosh is even and sinh is odd.

Also,

$$\cos z = \cosh(iz), \qquad i \sin z = \sinh(iz)$$

which show that cos and cosh are identical apart from a twist of 90 degrees in the variable.

Familiar identities like

$$\sin(z + w) = \sin z \cos w + \cos z \sin w, \qquad \sin^2 z + \cos^2 z = 1$$

can easily be proved.

DANGER: the second identity does NOT imply that $|\sin z| \leq 1$ and $|\cos z| \leq 1$ for all $z \in \mathbb{C}$. In fact, both $\sin$ and $\cos$ are unbounded on $\mathbb{C}$.

For example, let $z = iy$, where $y \in \mathbb{R}$. Then $|\cos z| = |\cos iy| = |\cosh y| = \cosh y \to \infty$ as $y \to \infty$. Thus $|\cos z|$ is unbounded on $\mathbb{C}$.

2.3 Examples

1. Find M such that $\left|\dfrac{e^z + \cos z}{6 + z}\right| \leq M$ for all z with $|z| = 1$.
2. Find the zeros of $\cos z$.

Solutions

1. Let $z = x + iy$. Then

$$\begin{aligned} |\cos z| &= \left|\tfrac{1}{2}\,(e^{iz} + e^{-iz})\right| &\leq \tfrac{1}{2}\,|e^{iz}| + \tfrac{1}{2}\,|e^{-iz}| \\ &= \tfrac{1}{2}\,e^{\mathrm{Re}(iz)} + \tfrac{1}{2}\,e^{\mathrm{Re}(-iz)} &= \tfrac{1}{2}\,e^{-y} + \tfrac{1}{2}\,e^{y} \quad = \quad \cosh y\,. \end{aligned}$$

Hence, for all z with $|z| = 1$,

$$|\cos z| \quad \leq \quad \cosh y \quad \leq \quad \cosh 1\,.$$

Using the triangle inequalities, we see that for $|z| = 1$,

$$|e^z + \cos z| \;\leq\; |e^z| + |\cos z| \;\leq\; e^{\,\mathrm{Re}\, z} + \cosh y \;\leq\; e^1 + \cosh 1.$$

and

$$|6+z| \;\geq\; |6|-|z| = 5 \qquad \text{giving} \quad \left|\frac{1}{6+z}\right| = \frac{1}{|6+z|} \;\leq\; \frac{1}{5}\,.$$

Hence

$$\left|\frac{e^z+\cos z}{6+z}\right| \;\leq\; \frac{e+\cosh 1}{5}\,,$$

for all z such that $|z|=1$.

2. Put $z+x+iy$. Then

$$\cos z \;=\; \cos(x+iy) \;=\; \cos x\cos(iy)-\sin x\sin(iy) \;=\; \cos x\cosh y - i\sin x\sinh y$$

and

$$\begin{aligned} |\cos z|^2 &= \cos^2 x\cosh^2 y+\sin^2 x\sinh^2 y \\ &= \cos^2 x(1+\sinh^2 y)+(1-\cos^2 x)\sinh^2 y \\ &= \cos^2 x+\sinh^2 y \end{aligned}$$

$\cos z = 0$, implies that

$$\cos^2 x+\sinh^2 y = 0 \quad \text{i.e.}\;\; \cos x = 0 = \sinh y\,,$$

since x and y are real numbers. Hence $x=\frac{\pi}{2}+n\pi$ and $y=0$. Thus the zeros of $\cos z$ are at $z=\frac{\pi}{2}+n\pi \quad (n=0,\,\pm1,\,\pm2,\cdots)$ i.e the only zeros of $\cos z$ are the familiar ones on the real axis.

Further, as $\cos(iz)=\cosh z$, for all complex numbers z, we deduce that the zeros of $\cosh z$ are at $\frac{i\pi}{2}+in\pi \quad (n=0,\,\pm1,\,\pm2,\cdots)$. Thus all the zeros of $\cosh z$ are on the imaginary axis.

Definition 2.3 *The Remaining Trigonometric and Hyperbolic functions are defined by:*

$$\begin{aligned} \tan z &= \tfrac{\sin z}{\cos z}\ \left(z\neq \tfrac{(2n+1)\pi}{2}\right), & \cot z &= \tfrac{\cos z}{\sin z}\ (z\neq n\pi), \\ \sec z &= \tfrac{1}{\cos z}\ \left(z\neq \tfrac{(2n+1)\pi}{2}\right), & \operatorname{cosec} z &= \tfrac{1}{\sin z}\ (z\neq n\pi), \\ \tanh z &= \tfrac{\sinh z}{\cosh z}\ \left(z\neq \tfrac{(2n+1)\pi i}{2}\right), & \coth z &= \tfrac{\cosh z}{\sinh z}\ (z\neq n\pi i), \\ \operatorname{sech} z &= \tfrac{1}{\cosh z}\ \left(z\neq \tfrac{(2n+1)\pi i}{2}\right), & \operatorname{cosech} z &= \tfrac{1}{\sinh z}\ (z\neq n\pi i). \end{aligned}$$

2.4 Complex logarithm

The exponential function e^x is a strictly increasing positive function on $\mathbb{R}$ and so we can define an inverse function $\ln x$ on $(0,\infty)$. Thus, for all positive real numbers x , we have $e^{\ln x}=x$.

Now let us see if we extend this idea to define the complex logarithm. Suppose $z\neq 0$. Any complex number w such that $e^w=z$ is defined to be a value of $\log z$. Then for all integers n, $e^{w+2n\pi i} = e^w = z$ and we see that if w is a value of $\log z$ then $w + 2n\pi i$ is also a value of $\log z$ for all integers n. Thus if $\log z$ has a value, then it has an infinite number of values and $\log z$ **is <u>NOT</u> a function.**

For $z \neq 0$ write $z = re^{i\theta}$ with $r > 0$ and write $\log z = w = u + iv$. Then $e^w = z$ and so

$$re^{i\theta} = z = e^w = e^u(\cos v + i \sin v)\,.$$

Hence $r = |z| = |e^u(\cos v + i\sin v)| = e^u$. Thus $\mathrm{Re}\,(\log z) = u = \ln |z|$. We also see that the values of v are $\theta + 2n\pi$. Thus the possible values of $\mathrm{Im}\,(\log z)$ are $\theta + 2n\pi$.

Thus, if $z \neq 0$ and $z = re^{i\theta}$ with $r > 0$, then

$$\log z = \ln r + i\theta + 2n\pi i = \ln |z| + i \arg z,$$

where $\arg z$ has infinitely many values, so that $\log z$ also has infinitely many values and any pair differ by an integral multiple of $2\pi i$.

Since $e^w \neq 0$ for all complex numbers w, we see that $\log z$ is not defined when $z = 0$.

(Note that, for $z \neq 0$ we use $\log z$ for the complex logarithm and $\ln |z|$ for the real logarithm of the positive real number $|z|$. Hence ln is defined on $\mathbb{R}^+$ and $\log z$ is defined on $\mathbb{C}\backslash\{0\}$.)

2.5 Examples

1. Find all the values of

   (i) $\log(-e)$, (ii) $\log(\sqrt{3} - i)$.

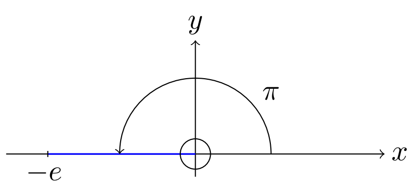
2. Find all the roots of the equation $e^{2z} + 1 + i = 0$.

3. Find all roots of the equation $\cosh z = -i$.

Solutions

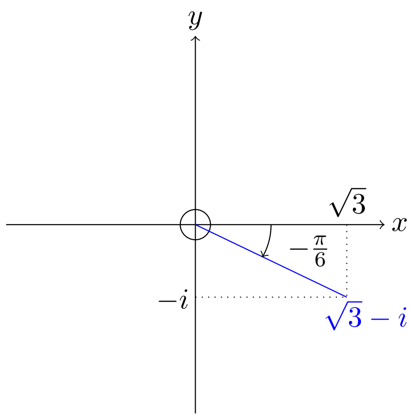
1. (i) In modulus-argument form $-e = e(\cos \pi + i \sin \pi)$ and so a value of $\log(-e)$ is $\ln | - e| + i \arg(-e) = 1 + i\pi$. All the values of $\log(-e)$ are therefore given by

$$1 + i(\pi + 2n\pi) \qquad (n \in \mathbb{Z})\,.$$



(ii) In modulus-argument form $\sqrt{3} - i = 2[\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6})]$. A value of $\log(\sqrt{3} - i)$ is, therefore, $\ln 2 - \frac{i\pi}{6}$. All values of $\log(\sqrt{3} - i)$ are given by

$$\ln 2 - \tfrac{i\pi}{6} + 2n\pi i\,.$$



2. Since $e^{2z} + 1 + i = 0$, we see that

$$e^{2z} = -1 - i\,.$$

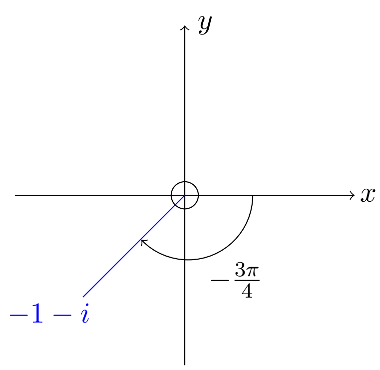
Thus the required values of $2z$ are the values of $\log(-1 - i)$

Now the values of $\log(-1 - i)$ are

$$\ln\sqrt{2}\; - \frac{3\pi i}{4} + 2n\pi i \qquad (n = 0, \pm 1, \pm 2, \cdots)$$

and so the required solutions are

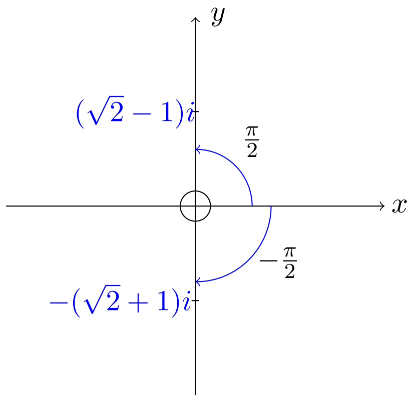
$$z = \tfrac{1}{2}\ln\sqrt{2}\; - \tfrac{3\pi i}{8} + n\pi i \qquad (n = 0, \pm 1, \pm 2, \cdots)\,.$$



3. Using $\cosh z = \frac{1}{2}(e^z + e^{-z})$, the equation $\cosh z = -i$ becomes

$$e^z + e^{-z} = -2i \qquad \text{i.e.} \quad e^{2z} + 2ie^z + 1 = 0.$$

This is a quadratic equation in e^z with roots $e^z = -i \pm \sqrt{i^2 - 1} = -i \pm i\sqrt{2}$, by the quadratic formula. Thus the required values of z are the values of $\log(-i \pm i\sqrt{2})$.



Now

$$\log(-i - i\sqrt{2}) \text{ has values } \ln(\sqrt{2}+1) - \tfrac{\pi i}{2} + 2n\pi i \quad (n \in \mathbb{Z}),$$

and

$$\begin{aligned} \log(-i + i\sqrt{2}) \quad \text{has values} \quad & \ln(\sqrt{2}-1) + \tfrac{\pi i}{2} + 2n\pi i \\ & = -\ln(\sqrt{2}+1) + \tfrac{\pi i}{2} + 2n\pi i \quad (n \in \mathbb{Z}) \end{aligned}$$

(since $\ln(\sqrt{2}-1) = -\ln(\sqrt{2}+1)$ because $(\sqrt{2}-1) = 1/(\sqrt{2}+1)$.)

Thus the required solutions are

$$z = \pm\left[\ln(\sqrt{2}+1) - \tfrac{\pi i}{2} + 2n\pi i\right] \qquad (n \in \mathbb{Z}).$$

3 Simple integrals of complex-valued functions

We want to develop calculus for complex-valued functions. Surprisingly, it easier to begin with integration rather than differentiation. So that is what we will do.
In the second year you met line integrals for functions of two variables and it is only a very short step from this to integrals for complex-valued functions. Of course we will need to be careful about the choice of curve over which to take the integral. Hopefully, the section below is really only a reminder of ideas you met in the second year.

3.1 Types of curves

A curve in the plane can be defined parametrically by $x = x(t)$, $y = y(t)$, $(a \le t \le b)$, where x and y have continuous derivatives x' and y' on $[a, b]$. Write $z = x + iy$ and $z(t) = x(t) + iy(t) \quad (a \le t \le b)$. Then we define $z'(t)$ to be $x'(t) + iy'(t)$ for $(a \le t \le b)$. For example,

$$(e^{it})' = (\cos t + i\sin t)' = -\sin t + i\cos t = i(\cos t + i\sin t) = ie^{it}$$

as expected.

We say that z' **is continuous on** $[a, b]$ **(or we say that** $z(t)$ **is continuously differentiable on** $[a, b]$**) when** x' **and** y' **are both continuous on** $[a, b]$.

Definition 3.1 A **curve** γ is defined by a continuously differentiable complex-valued function z of a real variable t on $[a,b]$ $(a,b \in \mathbb{R})$. So we write

$$\gamma : z = z(t) \qquad (a \le t \le b). \qquad \square$$

A curve has orientation defined by the parametrization and different parametrizations can produce the same oriented subset of $\mathbb{C}$. **The direction of the curve γ is the direction in which the parameter t increases so it goes from $z(a) = x(a) + iy(a)$ to $z(b) = x(b) + iy(b)$.** I will, therefore put an arrow to denote direction on a curve, whenever I draw a diagram containing one.

Our curves have a continuously turning tangent and we also suppose there are one-sided tangents at each of the end points.

A curve will have length $\int_a^b |z'(t)|\,dt$ which is finite as $|z'|$ is continuous on $[a,b]$.

Recall that an element of arc length $\delta s = [(\delta x)^2 + (\delta y)^2]^{\frac{1}{2}}$ and hence the length of γ is

$$\int_\gamma \delta s = \int_a^b \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \right]^{\frac{1}{2}} dt = \int_a^b |z'(t)|\,dt\,.$$

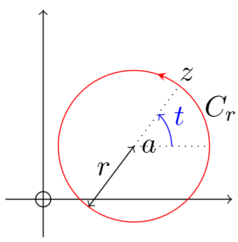
Definition 3.2 A **path** is a finite union of curves (joined successively at end points). A **contour** is a path whose final point is the same as its initial point. A **simple contour** is a contour without self-intersections.

Examples of paths and contours.

1. If z lies on a circle centre a and radius r, then $|z-a| = r$ i.e. $z-a$ can be written in modulus-argument form as $z - a = re^{it}$ for some $t \in \mathbb{R}$, giving $z = a + re^{it}$.

   Thus the circle C_r, with centre a and radius r, described in the anticlockwise direction can be given by

   $$z = a + re^{it} \quad (0 \le t \le 2\pi).$$



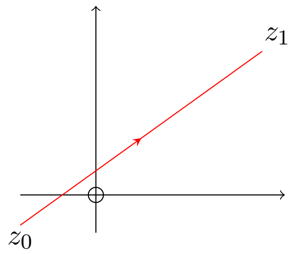
C_r is a **simple** contour

2. Suppose $z_0 \neq z_1$. The straight line segment from z_0 to z_1 can be given by

$$z = z(t) = tz_1 + (1-t)z_0 \quad (0 \le t \le 1). \tag{1}$$

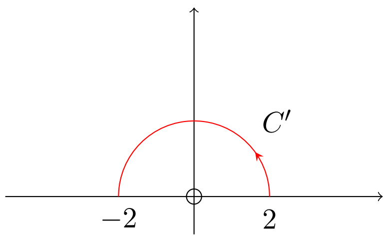
For, clearly the expression on the RHS of (1) is linear in t and so describes a straight line segment L. Moreover L goes from $z(0) = z_0$ to $z(1) = z_1$.

Note. Relation (1) can be rearranged to give $z = z(t) = z_0 + t(z_1 - z_0)$ $(0 \leq t \leq 1)$.

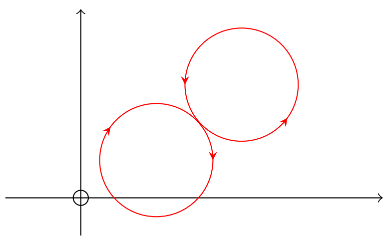


L is a path but it is not a contour

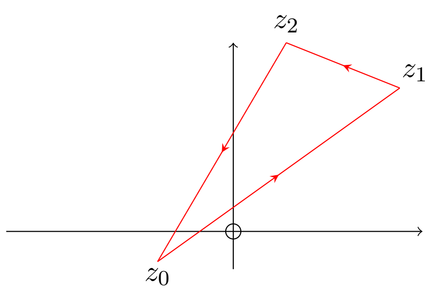
3. (i) The semi-circle given by $z = 2e^{it}$ $(0 \leq t \leq \pi)$ (shown below) is a path but it is not a contour.



(ii) The figure of eight shown below is a contour, but it is **not** a simple contour.



(iii) The triangular contour shown below is a simple contour.



3.2 Line integrals

Definition 3.3 Let f be continuous on a region containing the path γ. Let γ be given by $z = z(t)$, $a \leq t \leq b$. Then

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt. \qquad \square$$

Notes.

1. On $[a, b]$, $f(z(t))z'(t)$ is a continuous complex valued function of a real variable t, except for the finitely many values of t corresponding to corners. This is enough to show that the integral exists. (No proof.)

2. Whichever parametrization of γ is taken, you get the same value of $\int_\gamma f(z)dz$. (No proof.)

3. Let γ be a path. Then we define the path $-\gamma$ to be the path γ described with the opposite orientation (i.e. $-\gamma$ is the path γ taken in the opposite direction). Thus

$$\int_{-\gamma} f(z)dz = -\int_\gamma f(z)dz\,.$$

We assume this, and also easy results like:

$$\begin{aligned}
\int_\gamma (f_1(z) + f_2(z))dz &= \int_\gamma f_1(z)dz + \int_\gamma f_2(z)dz \\
\int_\gamma (cf(z))dz &= c\int_\gamma f(z)dz \qquad (c \in \mathbb{C}) \\
\int_{\gamma_1+\gamma_2} f(z)dz &= \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz,
\end{aligned}$$

where we use $\gamma_1 + \gamma_2$ to mean the path γ_1 followed by the path γ_2.

3.3 Examples

1. Evaluate $\int_\gamma z dz$ where γ is the union of the paths

$$\gamma_1 : z = x \quad (0 \leq x \leq 1) \quad \text{ and } \quad \gamma_2 : z = 1 + iy \quad (0 \leq y \leq 1).$$

2. Let $R > 0$ and $\gamma : z = Re^{it} \quad (0 \leq t \leq 2\pi)$. Let k be any integer with $k \geq 2$. Evaluate

$$\text{(i)} \int_\gamma \frac{1}{z}\,dz \qquad \text{(ii)} \int_\gamma \frac{1}{z^k}\,dz.$$

3. Evaluate $\int_\gamma \overline{z} dz$, where $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ where

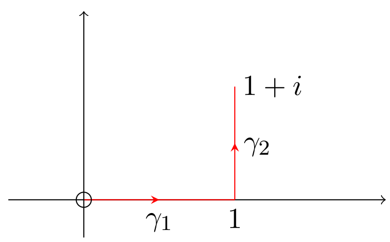
$$\begin{aligned} \gamma_1 : z &= x \qquad (-1 \le x \le 1) \\ \gamma_2 : z &= e^{it} \qquad (0 \le t \le \pi/2) \\ \gamma_3 : z &= i + (-1 - i)t \qquad (0 \le t \le 1) \end{aligned}$$

(Note: if a parametrisation is needed for a line segment from a point z_1 to the origin, it is often easier to parametrise the line segment in the opposite direction, from 0 to z_1, and then to use the relation $\int_\gamma = -\int_{-\gamma}$.)

Solutions

1. Since $\gamma = \gamma_1 + \gamma_2$, where

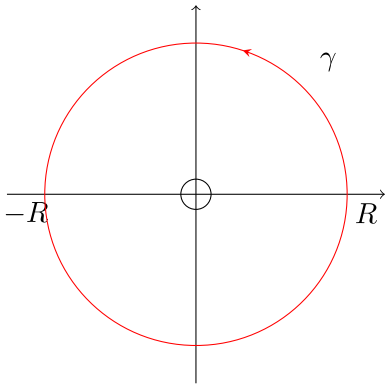
$$\gamma_1 : z = x \qquad (0 \le x \le 1) \quad \text{and} \quad \gamma_2 : z = 1 + iy \qquad (0 \le y \le 1),$$



and z is continuous on γ we see that

$$\int_\gamma z dz = \int_{\gamma_1} z dz + \int_{\gamma_2} z dz = \int_0^1 x dx + \int_0^1 (1 + iy) i dy = \tfrac{1}{2} + \left(i - \tfrac{1}{2}\right) = i.$$

2. (i) The curve γ is given by $z = Re^{it} \quad (0 \le t \le 2\pi)$.



Hence

$$\int_\gamma \frac{1}{z}\, dz = \int_0^{2\pi} \frac{1}{Re^{it}}\, Rie^{it} dt = 2\pi i.$$

(Note that $\frac{d}{dt}(e^{it}) = ie^{it}$.)

(ii) Since k is an integer with $k \ge 2$,

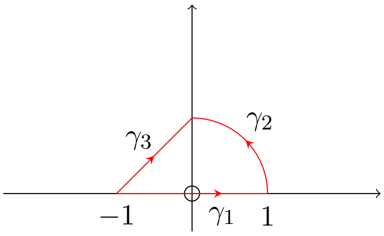
$$\int_\gamma \frac{1}{z^k} dz = \int_0^{2\pi} \frac{1}{R^k e^{ikt}} Rie^{it} dt = \frac{1}{R^{k-1}} \int_0^{2\pi} ie^{-(k-1)it} dt = \frac{1}{R^{k-1}} \left[\frac{ie^{-(k-1)it}}{-i(k-1)} \right]_{t=0}^{t=2\pi} = 0$$

as $e^{-(k-1)i2\pi} = e^0$ and $\int e^{iat} dt = \frac{e^{iat}}{ia}$.

Note the answer is independent of both R and k.

3. Now $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ where

$$\begin{aligned} \gamma_1 : z &= x \qquad (-1 \le x \le 1), \\ \gamma_2 : z &= e^{it} \qquad (0 \le t \le \tfrac{\pi}{2}), \\ \gamma_3 : z &= i + (-1-i)t \qquad (0 \le t \le 1) \end{aligned}$$



and $\overline{z}$ is continuous on γ.

Hence

$$\begin{aligned} \int_{\gamma_1} \overline{z} dz &= \int_{-1}^{1} x dx = 0\,, \\ \int_{\gamma_2} \overline{z} dz &= \int_0^{\frac{\pi}{2}} e^{-it} \tfrac{dz}{dt}\, dt = \int_0^{\frac{\pi}{2}} e^{-it} i e^{it}\, dt \;=\; i\tfrac{\pi}{2}\,, \\ \int_{\gamma_3} \overline{z} dz &= \int_0^1 (-i-t+it)(-1-i)\, dt \;=\; -(1+i)\int_0^1 (-i-t+it) dt \;=\; i \end{aligned}$$

and therefore,

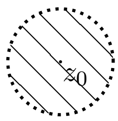
$$\begin{aligned} \int_\gamma \overline{z} dz &= \int_{\gamma_1} \overline{z} dz + \int_{\gamma_2} \overline{z} dz + \int_{\gamma_3} \overline{z} dz \\ &= 0 + i\tfrac{\pi}{2} + i = i\left(\tfrac{\pi}{2} + 1\right)\,. \end{aligned}$$

(Note: to parametrise a line segment from a point z_0 to the origin, it is often easier to parametrise the line segment in the opposite direction, from the origin to the point z_0, and then to use the relation $\int_\gamma = -\int_{-\gamma}$.)

4 Definitions about sets

Before defining the concept of a derivative, we need to say what type of sets we are going to use.

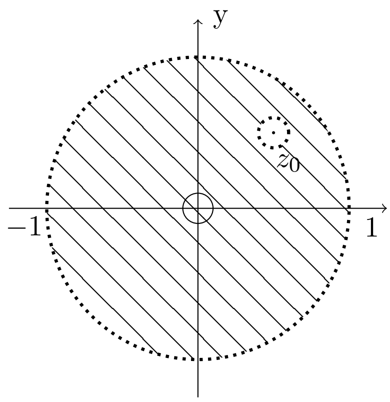
Definition 4.1 *A neighbourhood of $z_0 \in \mathbb{C}$ is a set of the form $\{z \in \mathbb{C} : |z - z_0| < \delta\}$ for some $\delta > 0$. (i.e. it is an open disc about z_0.)* □



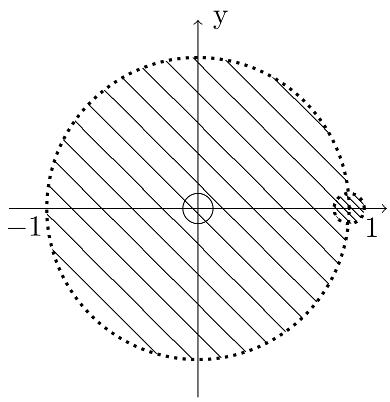
Definition 4.2 *A set $D \subseteq \mathbb{C}$ is said to be open if each point $z_0 \in D$ has a neighbourhood contained in the set. (A set is open if all its boundary points are "missing".)* $\square$

Examples.

- The disc $D = \{z \in \mathbb{C} : |z| < 1\}$ is open, since $D_{z_0} = \{z \in \mathbb{C} : |z - z_0| < \frac{1}{2}(1 - |z_0|)\}$ is a neighbourhood of z_0 lying in $\{z \in \mathbb{C} : |z| < 1\}$ whenever $|z_0| < 1$.



- The set $\{z \in \mathbb{C} : |z| < 1\} \cup \{1\}$ is not open, since any neighbourhood of 1 contains points with modulus greater than 1. Such points are outside the set.



Definition 4.3 *A non-empty open set $D \subseteq \mathbb{C}$ is connected if given any points z, $w \in D$, we can find a path $\gamma \subset D$ with initial point z and final point w. (i.e. D is "all in one piece")* $\square$

Definition 4.4 *A non-empty, open, connected set is called a region.* $\square$

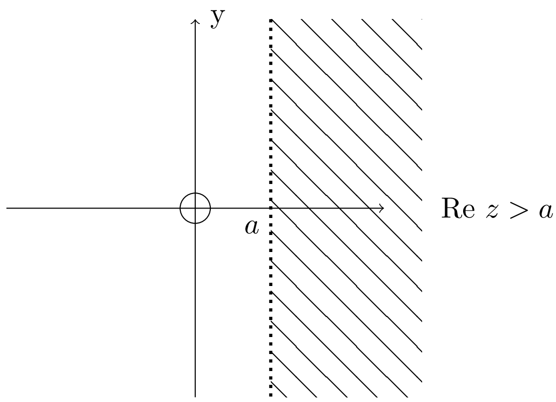
Definition 4.5 *A region D is said to be simply connected if it has no "holes", i.e., if every point in the interior of any simple contour in D is contained in D.* $\square$

<u>**Note**</u>**.** In order to decide whether a set is a region, you need to check whether the set is

(i) non-empty, (ii) open and (iii) connected.

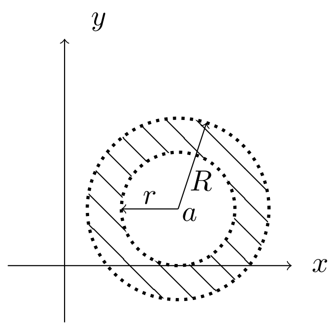
4.1 Examples

1. The half-plane $H = \{z \in \mathbb{C} : \operatorname{Re} z > a\}$ is a non-empty, open, connected set and it is also simply connected.



H is a simply connected region.

2. The annulus $A = \{z : r < |z - a| < R\}$ (with $0 < r < R$) is a non-empty, open, connected set, but it is not simply connected since the set $\{z \in \mathbb{C} : |z - a| \leq r\}$ is a hole (more formally, if $r < t < R$, the points in the interior of $\{z : |z - a| = t\}$ are all the points with $|z - a| < t$, but not all of these points are contained in the annulus itself).

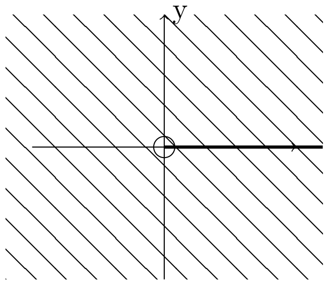


A is a region, **but it is not a simply connected region.**

3. The punctured plane $\mathbb{C} \setminus \{a\}$ is a non-empty, open, connected set but it is not simply connected (as it is easy to write down a contour with a in its interior, but a is not in the original set).

The punctured plane is a region, **but it is not a simply connected region.**

4. The cut plane $C^* = \mathbb{C} \setminus \{z \in \mathbb{C} : z \text{ is real and } z \geq 0\}$ is a non-empty, open, connected set and it is also simply-connected.



The cut plane C^* is a simply-connected region.

5 Differentiation

5.1 Limit

Definition 5.1 Let f be defined on some punctured neighbourhood of z_0 [i.e., it is defined on $\{z \in \mathbb{C} : 0 < |z - z_0| < \delta\}$ for some $\delta > 0$]. We use the phrase "$f(z) \to l$ as $z \to z_0$" to mean that $|f(z) - l| \to 0$ as $|z - z_0| \to 0$. □

Thus $f(z) \to l$ as $z \to z_0$ along any path approaching z_0 and the limit l, does not depend on the path chosen. Since $|f(z) - l|$ and $|z - z_0|$ are real valued no new principles are involved.

Note that we are demanding that a 2-dimensional limit exists, i.e., the limit exists in all directions towards z_0.

For example, $\dfrac{|z|}{z}$ has no limit as $z \to 0$, as $\dfrac{|z|}{z} = 1$ for z real and positive, and -1 for z real and negative. Hence there is no l such that $\dfrac{|z|}{z} \to l$ as $z \to 0$.

Note that to prove that a limit does not exist, it is sufficient to find 2 paths going to z_0 along which the limits exist and are not equal.

In view of the inequalities

$$\left.\begin{array}{l} |u - u_0| \\ \\ |v - v_0| \end{array}\right. \leq |w - w_0| = |u + iv - (u_0 + iv_0)| \leq |u - u_0| + |v - v_0|$$

we see that if $w = u + iv$ and $w_0 = u_0 + iv_0$ then $w \to w_0$ as $z \to z_0$ if and only if $u \to u_0$ **and** $v \to v_0$ as $z \to z_0$.

Continuity.

Definition 5.2 If f is defined on a neighbourhood (no longer punctured) of z_0 and $f(z) \to f(z_0)$ as $z \to z_0$, then f is said to be *continuous* at z_0. $\square$

In complex function theory, continuity is not really important in its own right. It normally occurs as a consequence of differentiability. The concept of differentiability is central to this course.

5.2 Differentiation

Definition 5.3 Let f be a complex valued function of the complex variable z. Suppose that f is defined on a neighbourhood of z_0. We say that f is *differentiable* at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. We call the limit the derivative of f at z_0 and denote it by $f'(z_0)$.

Definition 5.4 The function f is said to be *analytic* on a region D if f is differentiable at all points of D.

So far, we have only defined *analytic* for regions (which, by definition, are **open** sets).

Definition 5.5 *If z_0 is a point, we say that f is analytic at z_0 if f is analytic on some region containing z_0.* $\square$

Thus the statement that f is *analytic* at z_0 means that f is not only differentiable at z_0, but it is also differentiable at all points within some open disc with non-zero radius centred at z_0 i.e. f is differentiable at z_0 and also at all points sufficiently close to z_0.

Open sets are the natural domains of analytic functions since a function must be defined on a neighbourhood of each point of D.

The demand that a 2-dimensional limit exists means that some apparently innocent functions may not be differentiable. Fortunately, however, some familiar results continue to hold.

Familiar results from $\mathbb{R}$ which are true in $\mathbb{C}$:

- differentiability implies continuity;
- the sum and product of two differentiable functions is differentiable and the familiar rules for derivatives of sums and products continue to hold;
- for all non-negative integers n, z^n is differentiable and its derivative is nz^{n-1} ;

- an analytic function $f(w(z))$ of an analytic function is analytic on some region and the chain rule $\dfrac{df}{dz} = \dfrac{df}{dw}\dfrac{dw}{dz}$ holds. (This will be assumed without proof.)
- if the functions f and g are both differentiable at the point z_0 **AND** $g(z_0) \neq 0$, then the quotient $\frac{f(z)}{g(z)}$ is differentiable at z_0 and its derivative at z_0 is

$$\frac{f'(z_0)\, g(z_0) - f(z_0)\, g'(z_0)}{[g(z_0)]^2}.$$

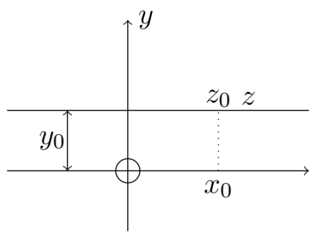
  If $g(z_0) = 0$, then the quotient $\frac{f(z)}{g(z)}$ is not defined at z_0 and the quotient is **NOT** differentiable at z_0.

  Note. Analytic functions are central to Complex Analysis. In order to find where functions involving quotients are analytic, you will need to find, first of all, where they are differentiable. You will frequently find that above will help you decide. So DON'T forget it.

The definition of differentiability for complex functions resembles that for real functions. However the consequences of differentiability for complex functions are far reaching as we will see in later sections. Analytic functions are very, very well behaved indeed. For example, it is not enough to ask for their real and imaginary parts to be "nice functions".

Example. Let $f(z) = \operatorname{Re} z$ i.e. $f(z) = f(x+iy) = x$. Then there is no point of $\mathbb{C}$ at which f is differentiable.

Solution. Firstly, let $z \to z_0$ along a line parallel to the real axis.



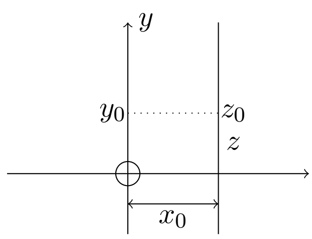
Write $z_0 = x_0 + iy_0$ and $z = x + iy_0$, where $x \neq x_0$. Then

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{x - x_0}{x - x_0} = 1$$

Thus, as $z \to z_0$ along a line parallel to the real axis,

$$\frac{f(z) - f(z_0)}{z - z_0} \to 1. \tag{1}$$

Secondly, let $z = x_0 + iy \to z_0$ along a line parallel to the imaginary axis.



Write $z_0 = x_0 + iy_0$ and $z = x_0 + iy$, where $y \neq y_0$. Then

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{x_0 - x_0}{i(y - y_0)} = 0$$

Thus, as $z \to z_0$ along a line parallel to the imaginary axis

$$\frac{f(z) - f(z_0)}{z - z_0} \to 0\,. \tag{2}$$

From (1) and (2), we see that

$$\frac{f(z) - f(z_0)}{z - z_0}$$

does not tend to any limit as $z \to z_0$. Hence the function f is not differentiable at z_0. This is true for all points $z_0 \in \mathbb{C}$.

Despite its innocent appearance, this is an example of a function which is continuous everywhere but nowhere differentiable.

It can **not** be differentiated because its real and imaginary parts are not related in the correct way. In fact they must be related by the Cauchy-Riemann equations if a function is to be differentiable.

5.3 The Cauchy-Riemann Equations

(Cauchy (1789–1857) and Riemann (1826–1866).)

Theorem 5.6 (Cauchy-Riemann Equations) *Let $f(z) = f(x+iy) = u(x,y)+iv(x,y)$, where u and v are real valued functions, and let $z_0 = x_0 + iy_0$. If f is differentiable at z_0, then u and v satisfy the relations*

$$u_x = v_y, \qquad u_y = -v_x \tag{$*$}$$

at (x_0, y_0),

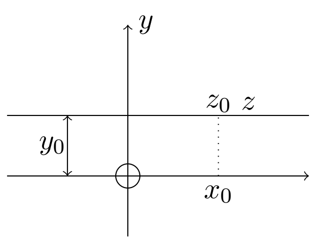
$$\text{i.e.} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{$*$}$$

at (x_0, y_0).
In the complex form this result becomes; "If the function f is differentiable at z_0, then $i\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$ at z_0." □

The relations (*) above are called the **Cauchy-Riemann Equations**.

Proof. As f is differentiable at z_0, f must be defined on a neighbourhood of z_0 and we can compute $\frac{df}{dz}$ at z_0 by letting $z \to z_0$ in any direction.

Firstly, let $z \to z_0$ (along a line parallel to the real axis).



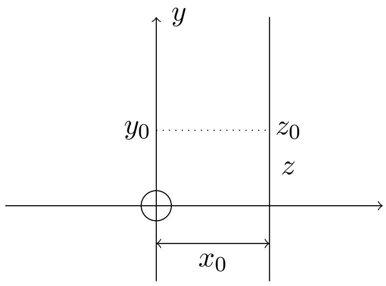
Write $z_0 = x_0 + iy_0$ and $z = x + iy_0$, where $x \neq x_0$. Then

$$\begin{aligned}
\frac{f(z) - f(z_0)}{z - z_0} &= \frac{[u(x,y_0) + iv(x,y_0)] - [u(x_0,y_0) + iv(x_0,y_0)]}{x - x_0} \\
&= \frac{u(x,y_0) - u(x_0,y_0)}{x - x_0} + i\,\frac{v(x,y_0) - v(x_0,y_0)}{x - x_0}\,. \\
&\to u_x(x_0,y_0) + iv_x(x_0,y_0) \quad \text{as } x \to x_0\,.
\end{aligned}$$

Thus

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = u_x(x_0,y_0) + iv_x(x_0,y_0)\,. \tag{1}$$

Secondly, let $z = x_0 + iy \to z_0$ (along a line parallel to the imaginary axis).



Write $z_0 = x_0 + iy_0$ and $z = x_0 + iy$, where $y \neq y_0$. Then

$$\begin{aligned}
\frac{f(z) - f(z_0)}{z - z_0} &= \frac{[u(x_0, y) + iv(x_0, y)] - [u(x_0, y_0) + iv(x_0, y_0)]}{i(y - y_0)} \\
&= \frac{u(x_0, y) - u(x_0, y_0)}{i(y - y_0)} + \frac{v(x_0, y) - v(x_0, y_0)}{y - y_0} . \\
&\to -iu_y(x_0, y_0) + v_y(x_0, y_0) \quad \text{as } y \to y_0 .
\end{aligned}$$

Thus

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = -iu_y(x_0, y_0) + v_y(x_0, y_0) . \qquad (2)$$

Equating real and imaginary parts in (1) and (2), gives

$$u_x(x_0, y_0) = v_y(x_0, y_0), \quad v_x(x_0, y_0) = -u_y(x_0, y_0)$$

$$\text{i.e. } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} , \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

at (x_0, y_0).

Equations (1) and (2) can be written as

$$\begin{aligned}
f'(z_0) &= [u_x + iv_x]_{x=x_0, y=y_0} &= \left[\tfrac{\partial f}{\partial x}\right]_{x=x_0, y=y_0} , \\
f'(z_0) &= [-iu_y + v_y]_{x=x_0, y=y_0} &= -i\left[\tfrac{\partial f}{\partial y}\right]_{x=x_0, y=y_0} .
\end{aligned}$$

Hence, at $z_0 = x_0 + iy_0$,

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y} .$$

This is the complex form of the Cauchy-Riemann equations. □

Notes:

1. The two forms of the Cauchy-Riemann equations are equivalent, as taking $i\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$, gives $i(u_x + iv_x) = u_y + iv_y$. Equating real and imaginary parts then gives (*).

2. There are four versions of f'. If $f = u + iv$ is analytic:

$$f'(z) = u_x + iv_x = u_x - iu_y = v_y - iu_y = v_y + iv_x.$$

3. Let $f(z) = \operatorname{Re} z = u + iv$, and let $z = x + iy$. Then $f(z) = f(x+iy) = x$ and so

$$u(x,y) = x \quad \text{and} \quad v(x,y) = 0\,.$$

Thus $u_x \neq v_y$ at all points of $\mathbb{C}$. Hence $\operatorname{Re} z$ is not analytic on $\mathbb{C}$ (in fact it is not differentiable at any point of $\mathbb{C}$). This is an example of a complex function which is continuous at every point of $\mathbb{C}$ and is not differentiable at any point.

4. The Cauchy-Riemann equations are axis dominated - they only make a statement about $\lim\limits_{z\to z_0} \dfrac{f(z)-f(z_0)}{z-z_)}$ as $z \to z_0$ in 2 directions, viz. along lines parallel to the real and imaginary axes. But we know that the derivative exists at z_0 if $\lim\limits_{z\to z_0} \dfrac{f(z)-f(z_0)}{z-z_0}$ exists for **all** approaches to z_0. Hence we cannot expect that the converse of this theorem will hold.

Indeed, there are examples in which the Cauchy-Riemann equations hold at a point z_0 but the function is **not** differentiable there.

Example. Let $z_0 = 0$, and let f be defined by

$$f(z) = \begin{cases} 1 & \text{on the axes} \\ 0 & \text{elsewhere.} \end{cases}$$

Write $f(z) = u(x,y) + iv(x,y)$, where u, v are real-valued so that

$$v(x,y) = 0, \quad u(x,0) = 1 \quad u(0,y) = 1$$

for all real x, y. Then

$$u_x(0,0) = \lim_{x\to 0}\left[\frac{u(x,0)-u(0,0)}{x}\right] = 0\,.$$

Similarly

$$u_y(0,0) = 0,\ v_x(0,0) = 0\ , v_y(0,0) = 0,$$

and the C-R equations are satisfied at the origin. But f is not even continuous at the origin and so it can not be differentiable there.

Thus the fact that the Cauchy-Riemann equations are satisfied at a certain point is NOT SUFFICIENT to guarantee differentiability at that point.

5.4 Examples

1. Prove that, f analytic on $\mathbb{C}$ and $\operatorname{Re} f$ constant on $\mathbb{C}$ implies that f is constant on $\mathbb{C}$.

2. Prove that, f analytic on $\mathbb{C}$ and $f' = 0$ on $\mathbb{C}$ implies that f is constant on $\mathbb{C}$.

3. Let f be analytic on $\mathbb{C}$ and suppose that $|f| = c$, a constant on $\mathbb{C}$. Prove that f is constant on $\mathbb{C}$.

Solutions.

1. We use the standard notation, $f(z) = f(x+iy) = u(x,y) + iv(x,y)$, where u, v are real-valued.

   In this example, we are told that $u = \operatorname{Re} f$ is constant. So

   $$u_x = u_y = 0$$

   everywhere. In addition, the function f is analytic in $\mathbb{C}$ and so the Cauchy-Riemann equations

   $$u_x = v_y \qquad v_x = -u_y$$

   are satisfied everywhere. Hence

   $$v_x = v_y = 0$$

   everywhere, and so v is independent of x and y. Thus v is constant, so $f = u + iv$ is constant.

2. Again use the standard notation and recall that there are four different forms for f' in terms of u and v. Since $f' = 0$ on $\mathbb{C}$. We see that

   $$0 = f' = u_x - iu_y = v_y + iv_x$$

   everywhere. Hence $u_x = u_y = v_x = v_y = 0$ everywhere and so u and v are constant, and so $f = u + iv$ is constant.

3. <u>If $c = 0$</u>, then clearly $f(z) = 0$ for all $z \in \mathbb{C}$.

   <u>If $c > 0$</u>, write $f = u + iv$. Then $|f(z)|^2 = [u(x,y)]^2 + [v(x,y)]^2 = c^2$. Taking partial derivatives with respect to x and y:

   $$\begin{aligned} 2uu_x + 2vv_x &= 0 \\ 2uu_y + 2vv_y &= 0\,. \end{aligned}$$

Since f is analytic in $\mathbb{C}$, u and v satisfy the C-R equations and so the above equations give,

$$\begin{aligned} uu_x - vu_y &= 0 \\ uu_y + vu_x &= 0\,. \end{aligned}$$

Eliminating u_y, we get

$$(u^2 + v^2)u_x^2 = 0, \quad \text{i.e. } c^2 u_x^2 = 0\,.$$

Hence $u_x = 0$ (as $c > 0$). Substitute back to find that $u_y = 0$. Thus $u = \mathrm{Re}(f)$ is constant, so f is constant by the first example.

5.5 Special Functions

We have seen that non-negative powers of z can be differentiated at every point of $\mathbb{C}$ and so polynomials are analytic in $\mathbb{C}$. It would really be rather dull if we had to restrict the rest of the course to polynomials. Fortunately, there are some more analytic functions readily available. Our problem is to prove that they can be differentiated. Remember that it is **NOT sufficient** to prove that the Cauchy Riemann equations are satisfied. We can, however, prove the following:

Theorem 5.7 *Let $f(z) = f(x + iy) = u(x, y) + iv(x, y)$, where u and v are real valued functions, and let $z_0 = x_0 + iy_0$. If*
(i) u and v have continuous first order partial derivatives at $(x_0,\, y_0)$
and
(ii) u and v satisfy the Cauchy-Riemann equations at $(x_0,\, y_0)$,
then f is differentiable at z_0.

Note. Condition (i) in the above theorem is too strong. It could be replaced by the condition that u and v are differentiable functions of two variables. This together with (ii) are both necessary and sufficient for differentiability of f.

Now let us apply this theorem to show that the exponential function is differentiable everywhere. Put $z = x + iy$, then

$$e^z = e^x(\cos y + i \sin y) = u(x, y) + iv(x, y)\,,$$

where

$$u(x, y) = e^x \cos y \quad \text{and} \quad v(x, y) = e^x \sin y\,.$$

Hence

$$u_x = e^x \cos y\,, \quad u_y = -e^x \sin y\,, \quad v_x = e^x \sin y\,, \quad v_y = e^x \cos y$$

and so the first order partial derivatives u_x, u_y, v_x, v_y are continuous and satisfy the Cauchy-Riemann equations $u_x = v_y,\ v_x = -u_y$ everywhere. The exponential function is, therefore, differentiable everywhere and

$$\frac{d}{dz}\left(e^z\right) = u_x + iv_x = e^x\cos y + ie^x\sin y = e^x(\cos y + i\sin y) = e^z\ . \qquad (1)$$

Thus the exponential function is analytic in $\mathbb{C}$.

Now, for all z,

$$\frac{d}{dz}\left(\cos z\right) = \frac{d}{dz}\left[\tfrac{1}{2}\left(e^{iz} + e^{-iz}\right)\right] = \tfrac{1}{2}\left(ie^{iz} - ie^{-iz}\right) = -\tfrac{1}{2i}\left(e^{iz} - e^{-iz}\right) = -\sin z\ .$$

Thus $\cos z$ is analytic in $\mathbb{C}$ and its derivative is $-\sin z$. Similarly $\sin z$, $\cosh z$, $\sinh z$ are all analytic in $\mathbb{C}$ and they have the familiar derivatives

$$\frac{d}{dz}\left(\sin z\right) = \cos z, \quad \frac{d}{dz}\left(\cosh z\right) = \sinh z, \quad \frac{d}{dz}\left(\sinh z\right) = \cosh z,$$

everywhere.

5.6 Harmonic functions

Definition 5.8 *A real valued function $u(x,y)$ on a region $D \subseteq \mathbb{R}^2$ which has*

1. *continuous second order partial derivatives, and*
2. *which satisfies Laplace's equation*

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

on D is said to be harmonic *on D.*

Theorem 5.9 *If $f = u + iv$ is analytic on a region D, then u and v are harmonic on D.* □

Proof. (Proper proof is Corollary 2 of Theorem 9.3) We assume that u_x, u_y, v_x and v_y have second order partial derivatives with respect to x and y and that $u_{xy} = u_{yx}$, $v_{xy} = v_{yx}$. Then, using the Cauchy-Riemann equations,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = \frac{\partial^2 v}{\partial x\partial y} - \frac{\partial^2 v}{\partial y\partial x} = 0$$

and

$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = -\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 u}{\partial y\partial x} = 0\,.$$

□

Question. Given a function u which is harmonic on a region D, can we find a function f analytic on D with $u = \operatorname{Re} f$?

Answer. If D is simply connected, then the answer is yes. (Proof omitted.)

Possible Methods for finding f when u is given.

First check that u is harmonic.

Then:

Method (a). Find v using $v_x = -u_y$ and $v_y = u_x$. Then form $u + iv$ and express it in terms of z only (i.e., not x and y, nor $\operatorname{Re} z$, $\overline{z}$, $|z|$, $\cdots$).

Method (b) [Preferable]. Form $u_x - iu_y$, and express in terms of z to get $f'(z)$, and then integrate to get f. (Remember that we found, in the section on the Cauchy-Riemann equations, that one form for $f'(z)$ is $u_x - iu_y$.)

The advantage of method (b) is that v is ignored.

5.7 Examples

In each of the following cases decide whether there is a function f analytic on $\mathbb{C}$ such that $\operatorname{Re} f = u$. When f exists, find an expression for $f(z)$ in terms of z.

1. $u(x, y) = \sin(x^2 + y^2)$ on $\mathbb{R}^2$.
2. $u(x, y) = \sinh x \cos y - \cosh x \sin y$ on $\mathbb{R}^2$.

Solutions.

1. Use the standard notation. In this case $u(x, y) = \sin(x^2 + y^2)$ on $\mathbb{C}$ and so

$$\begin{aligned}
\nabla^2 u &= \frac{\partial^2}{\partial x^2}[\sin(x^2+y^2)] + \frac{\partial^2}{\partial y^2}[\sin(x^2+y^2)] \\
&= \frac{\partial}{\partial x}[2x\cos(x^2+y^2)] + \frac{\partial}{\partial y}[2y\cos(x^2+y^2)] \\
&= 2\cos(x^2+y^2) - 4x^2\sin(x^2+y^2) + 2\cos(x^2+y^2) - 4y^2\sin(x^2+y^2) \\
&= 4\cos(x^2+y^2) - (4x^2+4y^2)\sin(x^2+y^2) \\
&\neq 0
\end{aligned}$$

So u is not harmonic, and so no f exists. $\square$

2. We have $u(x,y) = \sinh x \cos y - \cosh x \sin y$ on $\mathbb{R}^2$ and so

$$\begin{aligned} \nabla^2 u &= \frac{\partial^2}{\partial x^2}[\sinh x \cos y - \cosh x \sin y] + \frac{\partial^2}{\partial y^2}[\sinh x \cos y - \cosh x \sin y] \\ &= \frac{\partial}{\partial x}[\cosh x \cos y - \sinh x \sin y] + \frac{\partial}{\partial y}[-\sinh x \sin y - \cosh x \cos y] \\ &= \sinh x \cos y - \cosh x \sin y - \sinh x \cos y + \cosh x \sin y \\ &= 0 \end{aligned}$$

So u is harmonic. As $\mathbb{C}$ is simply connected, the stated result shows that f exists, analytic on $\mathbb{C}$ with $\operatorname{Re} f = u$. We use Method (b) to find f. We know that

$$\begin{aligned} f' &= u_x - iu_y \\ &= \cosh x \cos y - \sinh x \sin y + i \sinh x \sin y + i \cosh x \cos y \end{aligned}$$

which we need to express in terms of z only. It may not be immediately obvious how to do this. If we could even make a sensible guess as to the answer it would be straightforward to check whether the guess was valid or not. But an educated guess may be very hard to make. Perhaps what we need is a clever little trick to help us on our way!!

Clever little trick. If I can write down a formula giving $f'(x)$ in terms of x. Then this formula with every x replaced by z should, hopefully, at least provide a sensible guess for $f'(z)$ in terms of z.
Now I can obtain my formula for $f'(x)$ by taking my expression for $f'(x+iy)$ and substituting in $y = 0$.
If this method yields a result, I can eliminate any element of doubt by checking that the answer I obtain does, indeed, satisfy all the given conditions.

In this example,

$$f'(x + i0) = (1+i)\cosh x$$

So there's a chance that $f'(z) = (1+i)\cosh z$, and then $f(z) = (1+i)\sinh z + c$ for some constant c. Let's try that, and hope that $\operatorname{Re} f = u$.

Check: Using $f(z) = (1+i)\sinh z + c$ gives,

$$\begin{aligned} \operatorname{Re} f &= \operatorname{Re}[(1+i)\sinh(x+iy) + c] \\ &= \operatorname{Re}[(1+i)(\sinh x \cosh iy + \cosh x \sinh iy) + c] \\ &= \operatorname{Re}[(1+i)(\sinh x \cos y + i \cosh x \sin y) + c] \\ &= \sinh x \cos y - \cosh x \sin y + \operatorname{Re}(c) \\ &= u + \operatorname{Re}(c) \end{aligned}$$

So this f works, so long as the real part of c is 0, i.e. $f(z) = (1+i)\sinh z + ia$, where $a \in \mathbb{R}$.

In fact, all functions f on $\mathbb{C}$ with $\operatorname{Re} f = u$ are of the form $(1+i)\sinh z + ia$ $(a \in \mathbb{R})$ as the following theorem shows.

Theorem 5.10 *Suppose that f and g are analytic on a region D and $\operatorname{Re} f = \operatorname{Re} g$ on D. Then $f = g + ia$ $(a \in \mathbb{R})$.*

Proof. Write $h = f - g$. Then h is analytic on the region D. Using the standard notation, we write $h(z) = h(x+iy) = u(x,y) + iv(x,y)$. Hence

$$u(x,y) \;=\; \operatorname{Re}(h(x+iy)) \;=\; \operatorname{Re}(f(x+iy)) - \operatorname{Re}(g(x+iy)) \;=\; 0$$

on D. Using the Cauchy-Riemann equations (or one of our previous examples) we see that, v is constant. So $h = ia$ for some $a \in \mathbb{R}$ because $\operatorname{Re}(h(z)) = 0$ on D. $\square$

From now on, we treat f as the main function, and do not split into $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$. There will be more results on differentiation but these will be found using integration.

Analysis in $\mathbb{R}$ uses differentiation and integration separately, and unites them in the Fundamental Theorem of Calculus. There is, however, NOTHING SIMILAR IN COMPLEX ANALYSIS.

5.8 Examples

1. The function f is analytic in $\mathbb{C}$ and its real and imaginary parts u, v satisfy the relation $u = 1 + v$. Show that f is constant.

2. The function f is analytic in $\mathbb{C}$ and its real and imaginary parts u and v satisfy $ue^v = 12$ at all points of $\mathbb{C}$. Prove that f is constant.

Solutions.

1. Since $u(x,y) = 1 + v(x,y)$ for all real numbers x, y, differentiating with respect to x and with respect to y gives

   $$u_x = v_x \quad (1) \qquad\qquad u_y = v_y \quad (2)$$

   everywhere. Now f is analytic in $\mathbb{C}$ and so u and v satisfy the Cauchy-Riemann equations

   $$u_x = v_y \quad (3) \qquad\qquad u_y = -v_x \quad (4)$$

   everywhere. Using equations (1), (4), (2) and (3) gives

   $$u_x = v_x = -u_y = -v_y = -u_x$$

everywhere. Hence $u_x = 0$ everywhere and therefore,

$$u_x = u_y = 0 \quad \text{ and } v_x = v_y = 0$$

everywhere. Thus u and v are constant and so $f = u + iv$ is constant.

2. As $ue^v = 12$ everywhere, we differentiate with respect to x to get

$$u_x e^v + ue^v v_x = 0,$$

which we can rewrite as

$$u_x + uv_x = 0 \tag{5}$$

(as $e^v \neq 0$).
Now differentiate with respect to y:

$$u_y e^v + ue^v v_y = 0,$$

which we can rewrite as

$$u_y + uv_y = 0. \tag{6}$$

But f is analytic on $\mathbb{C}$, and so u and v satisfy the Cauchy-Riemann equations everywhere. So

$$u_x = v_y \quad (7) \qquad\qquad u_y = -v_x\,. \quad (8)$$

Substitute (8) and (7) into (6) and (5) to get

$$u_y + uu_x = 0 \tag{9}$$

$$u_x + u(-u_y) = 0\,. \tag{10}$$

From (9) and (10) we get

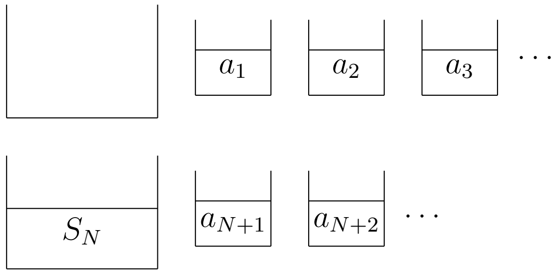
$$(1 + u^2)u_x = 0.$$

As u is real-valued, $1 + u^2 \geq 1$, and so $u_x = 0$ everywhere. Then $u_y = 0$ from (9). As $u_x = 0 = u_y$ everywhere, u is constant. Using the Cauchy-Riemann equations, $v_x = v_y = 0$ everywhere, so v is also constant. Thus $f = u + iv$ is constant.

6 Power series

6.1 Series with non-negative terms

We begin with a reminder of first year results. Suppose that a_1, a_2, $\cdots$ are non-negative real numbers. Pour volumes a_1, a_2, a_3, a_4, $\cdots$ of liquid into a vessel.



After N contributions, the vessel contains a volume $S_N = a_1 + a_2 + a_3 + a_4 + \cdots + a_N$.

There are two possibilities:
(i) $S_N \to \infty$ as $N \to \infty$. Then, no matter how large the vessel, the liquid will overflow eventually and the series $\sum_{n=1}^{\infty} a_n$ diverges.

(ii) The partial sums are bounded. Then the vessel will not overflow if it is large enough i.e. there exists some M such that $S_N \leq M$ for all N.

Fundamental Fact
In case (ii) there is a smallest vessel just large enough not to overflow. If its volume is L, then $S_N \to L$ as $N \to \infty$. The series converges and $\sum_{n=1}^{\infty} a_n = L$.

Notice that S_1, S_2, S_3, $\cdots$ is a set of numbers in a definite order. Such a set of numbers is called a sequence.
It is clear that the sequence of partial sums S_n is increasing whenever $a_n \geq 0$. The preceding result is a case of the general result:

Theorem 6.1 *An increasing sequence of real numbers is either*
(i) bounded above and tends to a finite limit
or
(ii) not bounded above and tends to infinity.

There is an example in the the first year notes where the above ideas are used to prove that $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by comparing its terms with those of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which is known to converge. Since

$$0 \leq \frac{1}{n^3} \leq \frac{1}{n^2} \qquad (n \geq 1)\,,$$

$$0 \leq T_N = \sum_{n=1}^{N} \frac{1}{n^3} \;\leq\; S_N = \sum_{n=1}^{N} \frac{1}{n^2}\,.$$

Now S_N tends to a finite limit as $N \to \infty$ and hence T_N also tends to a finite limit as $N \to \infty$ i.e. $\sum_{n=1}^{\infty} \frac{1}{n^3}$ also converges.

This is an illustration of the use of the Comparison Test.

Theorem 6.2 *Suppose that $a_n \geq 0, b_n \geq 0$ and that*

$$0 \leq b_n \leq a_n \quad \textit{for all } n \geq 1.$$

If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ also converges.

6.2 Series of complex numbers

The condition that the terms of the series are non-negative real numbers can be relaxed. Though the first year concentrated on series of real numbers, some of the results are true for real and complex series and your first year notes make this clear.
For the sake of clarity I normally use z_n rather than a_n for terms of complex series.

Definition 6.3 *Suppose that $z_n \in \mathbb{C}$ for all n and that $S_N = z_1 + z_2 + \cdots + z_N$. Then we say that $\sum_{n=1}^{\infty} z_n$ converges if there is a complex number L such that $S_N \to L$ as $N \to \infty$ i.e. there is a complex number L such that $|S_N - L| \to 0$ as $N \to \infty$.*

If no such complex number exists, then we say that the series diverges.

Note. It is easy to show that $S_N \to L$ as $N \to \infty$ if and only if $\mathrm{Re}\,(S_N) \to \mathrm{Re}\,L$ **and** $\mathrm{Im}\,(S_N) \to \mathrm{Im}\,L$ as $N \to \infty$.

Series of complex numbers were mentioned in the first year. Bearing in mind that it is a little time since your first year I felt that a reminder of some of the results might be valuable in case they are no longer fresh in your mind.

Theorem 6.4 *If $\sum_{n=1}^{\infty} z_n$ converges, then $z_n \to 0$ as $n \to \infty$.*

Note This Theorem guarantees the following:

$$\text{If } z_n \nrightarrow 0 \text{ as } n \to \infty, \text{ then } \sum_{n=1}^{\infty} z_n \textbf{ diverges}.$$

Reminder In the first year, you met the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \; .$$

This is an example of a divergent series $\sum_{n=1}^{\infty} z_n$ for which $z_n \to 0$ as $n \to \infty$.
Hence $z_n \to 0$ as $n \to \infty$ is NOT sufficient to guarantee the convergence of $\sum z_n$.
Despite persistent folklore in undergraduate circles this is still not sufficient this year !!

In the first year you also had the following result:

Definition 6.5 *A series $\sum_{n=1}^{\infty} z_n$ of complex numbers is said to be absolutely convergent if $\sum_{n=1}^{\infty} |z_n|$ converges.*

Theorem 6.6 *An absolutely convergent series of real or complex numbers is also convergent.*

Note For all complex numbers z_n, $|z_n|$ is a non-negative real number and so any result for series of non-negative real numbers is applicable to the series $\sum_{n=1}^{\infty} |z_n|$.

6.3 Power Series

Definition 6.7 *Suppose that $a_n \in \mathbb{C}$ for all n and $z_0 \in \mathbb{C}$. A series of the form $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is called a power series (centred on z_0).*

For the sake of simplicity we will put $z_0 = 0$. The generalization of the results to power series with other centres of expansion is obvious.

Examples.

1. $\sum_{n=0}^{\infty} n!z^n$ (here, $0! \equiv 1$).
2. $\sum_{n=0}^{\infty} z^n$.
3. $\sum_{n=0}^{\infty} \frac{z^n}{n!}$.

6.4 Convergence and absolute convergence

Theorem 6.8 *Suppose $w \neq 0$ and $\sum a_n w^n$ is convergent. Then $\sum a_n z^n$ is absolutely convergent for all z such that $|z| < |w|$.*

So if a power series converges at some point, it is absolutely convergent at any point nearer the origin.

Proof. If $\sum a_n w^n$ is convergent, then $a_n w^n \to 0$ as $n \to \infty$. As we explain below, this means that there is some M such that $|a_n w^n| \leq M$ for all n Take z with $|z| < |w|$. Then

$$|a_n z^n| = |a_n w^n| \left|\frac{z}{w}\right|^n \leq M \left|\frac{z}{w}\right|^n.$$

But $\sum M|\frac{z}{w}|^n$ is a convergent geometric progression, since $\left|\frac{z}{w}\right| < 1$. By the comparison test, $\sum |a_n z^n|$ is convergent, i.e., $\sum a_n z^n$ is absolutely convergent.

Note. Now we explain that $b_n \to 0$ implies that $|b_n| \leq M$ for some $M > 0$ and all n. As $b_n \to 0$, we can find some $N \in \mathbb{N}$ such that $|b_n| \leq 1$ for all $n \geq N$. Take $M = \max\{|b_1|, |b_2|, \ldots, |b_{N-1}|, 1\}$. This is a finite set of real numbers, so has a maximum. $\square$

6.5 Radius of convergence

Theorem 6.9 (Abel) *For the power series $\sum a_n z^n$, one of the following is true:*

1. *the power series converges only at $z = 0$ (e.g., $\sum n!z^n$)*
2. *the power series is absolutely convergent for all $z \in \mathbb{C}$ (e.g., $\sum \frac{z^n}{n!}$)*
3. *we can find a real number R with $0 < R < \infty$ such that the power series is absolutely convergent if $|z| < R$ and divergent if $|z| > R$.*

R is called the *radius of convergence* of the power series. In case (1), we put $R = 0$ and in case (2) we put $R = \infty$.

A formal proof of the above theorem can be found in any standard text on power series.

Definition 6.10 (Radius of Convergence, Disc of Convergence) *The quantity R in the above theorem (case (3)) is called the* radius of convergence *of the power series. In case (1), we put $R = 0$ and in case (2) we put $R = \infty$.*

In case (3) the disc $D = \{z \in \mathbb{C} : |z| < R\}$ is called the disc of convergence of $\sum a_n z^n$. In case (2) the disc of convergence is $\mathbb{C}$.

Note. Those of you who took the real analysis course will recognize that

$$R = \sup\{|z| : \sum |a_n z^n| \text{ converges.}\}$$

6.6 A possible formula for the radius of convergence

The formula which you had for the radius of convergence in the first year remains valid for complex power series.

Theorem 6.11 *If $|\frac{a_n}{a_{n+1}}| \to R$ as $n \to \infty$, then R is the radius of convergence of $\sum a_n z^n$.*

Proof. (i) If $z \neq 0$ and $R > 0$,

$$\left|\frac{a_{n+1} z^{n+1}}{a_n z^n}\right| = \left|\frac{a_{n+1}}{a_n}\right| |z| \to \frac{|z|}{R} < 1 \quad \text{as } n \to \infty$$

whenever $|z| < R$. Hence the power series converges absolutely for $|z| < R$.

(ii) If $z \neq 0$ and $R > 0$,

$$\left|\frac{a_{n+1} z^{n+1}}{a_n z^n}\right| = \left|\frac{a_{n+1}}{a_n}\right| |z| \to \frac{|z|}{R} > 1 \quad \text{as } n \to \infty$$

whenever $|z| > R$. Hence the power series diverges for $|z| > R$.

The result follows from (i) and (ii) when $R > 0$.
If $R = 0$ and $z \neq 0$,

$$\left|\frac{a_{n+1}z^{n+1}}{a_n z^n}\right| = \left|\frac{a_{n+1}}{a_n}\right| |z| \to \infty \quad \text{as } n \to \infty$$

and the ratio test shows that the power series is divergent for all $z \neq 0$. $\square$

Corollary 6.12 *If $|\frac{a_n}{a_{n+1}}| \to \infty$ as $n \to \infty$, then the power series $\sum a_n z^n$ has infinite radius of convergence i.e it converges absolutely for all z.*

A power series <u>always</u> has a radius of convergence irrespective of whether $\left|\frac{a_n}{a_{n+1}}\right|$ tends to a limit as $n \to \infty$ or not.

6.7 Examples

Find the radius of convergence of each of the following power series:

1. $\sum_0^\infty (\sinh n) z^n$; 2. $\sum_1^\infty \frac{2^n z^n}{n^2}$, 3. $\sum_1^\infty \frac{2^n z^{4n}}{n^2}$; 4. $\sum_1^\infty \frac{(3n)!\,n!}{(4n)!} z^n$.

Solutions. 1. Using $\sum_0^\infty (\sinh n) z^n = \sum_0^\infty a_n z^n$, we have,

$$\begin{aligned}\left|\frac{a_n}{a_{n+1}}\right| &= \left|\frac{\sinh n}{\sinh(n+1)}\right| = \frac{\sinh n}{\sinh(n+1)} = \frac{\frac{1}{2}\,(e^n - e^{-n})}{\frac{1}{2}\,(e^{n+1} - e^{-(n+1)})} \\ &= \frac{1 - e^{-2n}}{e - e^{-2n-1}} \to \frac{1-0}{e-0} = \frac{1}{e} \quad \text{as} \quad n \to \infty\,.\end{aligned}$$

It follows that the power series has radius of convergence $R = \frac{1}{e}$.

2. Using $\sum_1^\infty \frac{2^n z^n}{n^2} = \sum_1^\infty a_n z^n$, gives,

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{2^n (n+1)^2}{2^{n+1} n^2} = \frac{1}{2}\left(1 + \frac{1}{n}\right)^2 \to \frac{1}{2} \text{ as } n \to \infty\,.$$

So the power series has radius of convergence $R = \frac{1}{2}$, by Theorem 6.11.

3. We can not apply Theorem 6.11 directly to the power series $\sum_1^\infty \frac{2^n z^{4n}}{n^2}$, because the coefficient of z^n is zero unless n is a multiple of 4. In fact the power series is

$$0 + 0 \times z + 0 \times z^2 + 0 \times z^3 + 2z^4 + 0 \times z^5 + 0 \times z^6 + 0 \times z^7 + z^8 + \cdots\,.$$

So the ratio of the coefficients of successive powers of z is frequently undefined. To get round the problem, we do the following.

Replace z^4 by w, and consider the power series $\sum_{n=1}^\infty \frac{2^n w^n}{n^2} = \sum_{n=1}^\infty a_n w^n$, which has radius of convergence $R = \frac{1}{2}$ by the last example.

Thus $\sum_{n=1}^{\infty} \frac{2^n w^n}{n^2}$ is absolutely convergent if $|w| < \frac{1}{2}$ and is divergent if $|w| > \frac{1}{2}$.

It follows that $\sum_{n=1}^{\infty} \frac{2^n z^{4n}}{n^2}$ is absolutely convergent if $|z^4| < \frac{1}{2}$ and is divergent if $|z^4| > \frac{1}{2}$.

Hence $\sum_{n=1}^{\infty} \frac{2^n z^{4n}}{n^2}$ is absolutely convergent if $|z| < 2^{-\frac{1}{4}}$ and is divergent if $|z| > 2^{-\frac{1}{4}}$.

So the radius of convergence $R = 2^{-\frac{1}{4}}$.

4. Write $\sum_{n=1}^{\infty} \frac{(3n)!n!}{(4n)!} z^n = \sum_{n=1}^{\infty} a_n z^n$. Then

$$\begin{aligned}
\left|\frac{a_n}{a_{n+1}}\right| &= \frac{(3n)!}{(3(n+1))!}\frac{n!}{(n+1)!}\frac{(4(n+1))!}{(4n)!} \\
&= \frac{(4n+4)(4n+3)(4n+2)(4n+1)}{(3n+3)(3n+2)(3n+1)(n+1)} \\
&= \frac{4^4}{3^3}\frac{(1+\frac{1}{n})(1+\frac{3}{4n})(1+\frac{2}{4n})(1+\frac{1}{4n})}{(1+\frac{1}{n})(1+\frac{2}{3n})(1+\frac{1}{3n})(1+\frac{1}{n})} \\
&\to \frac{4^4}{3^3}
\end{aligned}$$

as $n \to \infty$. Thus the power series has radius of convergence $R = \frac{4^4}{3^3}$.

Theorem 6.13 *The power series $\sum_0^{\infty} a_n z^n$ and $\sum_1^{\infty} n a_n z^{n-1}$ both have the same radius of convergence.*

Note If $|\frac{a_n}{a_{n+1}}| \to R$ as $n \to \infty$, the result is easy to prove, as this implies that

$$\left|\frac{n a_n}{(n+1)a_{n+1}}\right| = \left|\frac{n}{(n+1)}\right| \left|\frac{a_n}{a_{n+1}}\right| \to R \text{ as } n \to \infty\,.$$

This makes the result plausible. A full proof can be found in any standard text on power series

From the above result, we see that $\sum_0^{\infty} a_n z^n$ and $\sum_1^{\infty} n a_n z^{n-1}$ both have the same disc of convergence.

Theorem 6.14 *Suppose that $\sum_0^{\infty} a_n z^n$ has radius of convergence R. Define f by*

$$f(z) = \sum_0^{\infty} a_n z^n \quad (|z| < R)\,.$$

Then the function f is differentiable on the disc of convergence $D = \{z \in \mathbb{C} : |z| < R\}$ and

$$f'(z) = \sum_1^{\infty} n a_n z^{n-1} \quad (|z| < R).$$

Thus a power series can be differentiated term by term inside the disc of convergence. Since the disc D is an open set on which f is differentiable, the function f is analytic on

D.

If $\sum_0^\infty a_n z^n$ has infinite radius of convergence, then the sum function f is analytic on $\mathbb{C}$. **Thus the sum function of a power series is analytic on the disc of convergence.**

It is also true that a power series can be integrated term by term in the disc of convergence. In fact power series can be handled in the same way as polynomials inside the disc of convergence.

6.8 Power series for Exponential, Trigonometric and Hyperbolic Functions

To get the power series for $\cos z$ about $z = 0$, we use the power series $e^z = \sum_{n=0}^\infty \frac{z^n}{n!}$ and the expression $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$. Thus

$$\cos z = \frac{1}{2}\sum_{n=0}^\infty \left(\frac{(iz)^n}{n!} + \frac{(-iz)^n}{n!}\right) = \frac{1}{2}\sum_{n=0}^\infty (i^n + (-i)^n)\frac{z^n}{n!}$$

If n is odd, then $i^n + (-i)^n = 0$. If $n = 2k$ is even, then $i^{2k} + (-i)^{2k} = (-1)^k \times 2$. Thus

$$\cos z = \sum_{k=0}^\infty \frac{(-1)^k}{(2k)!} z^{2k}$$

and the power series is valid for all $z \in \mathbb{C}$.

Similarly, we get the natural extensions of the real series for $\sin z$, $\cosh z$ and $\sinh z$.

6.9 Switching orders

There are three big assumptions, all valid in this course with our situation, but which are false in general.

A1. You can switch orders of summation and differentiation: i.e., if each function g_n $(n = 1, 2, \ldots)$ is analytic on a region D, and if for each $z \in D$ the sum $\sum_n g_n(z)$ is convergent, then $\sum_n g_n$ is analytic on D and $(\sum_n g_n)' = \sum_n g_n'$ on D.

A2. You can switch orders of summation and integration:

$$\sum\left(\int_\gamma g_n\right) = \int_\gamma \left(\sum_n g_n\right).$$

A3. You can switch the order of differentiation and integration:

$$\frac{d}{dz}\left(\int_\gamma g(z,w)dw\right) = \int_\gamma \frac{\partial}{\partial z} g(z,w)dw.$$

7 More on $\int_\gamma f$

7.1 Primitives

Definition 7.1 *Let f be a complex-valued function defined on a region D. A function g analytic on D and such that $g' = f$ at all points of D is said to be a primitive of f on D (i.e., a primitive is an indefinite integral).* $\square$

Thus, if γ is ANY path in D, given by $z = z(t)$ for $a \le t \le b$, then

$$\begin{aligned} \int_\gamma f(z)\,dz &= \int_\gamma g'(z)\,dz = \int_a^b g'(z(t))z'(t)\,dt = \int_a^b \tfrac{d}{dt}[g(z(t))]\,dt \\ &= [g(z(t))]_a^b = g(z(b)) - g(z(a)) = [\,g(z)\,]_\gamma\,. \end{aligned}$$

Note that $\frac{d}{dt}(g(z(t))) = g'(z(t))\frac{dz}{dt}$ and $[\,g(z)\,]_\gamma$ is used for the value of g at the final point of γ minus the value of g at the initial point of γ.

Note that if γ is a contour, then $z(a) = z(b)$, and so $\int_\gamma f(z)dz = 0$. So if γ is a contour in a region and f has a primitive in the region then $\int_\gamma f(z)dz = 0$.

7.2 Examples

1. Evaluate $\int_\gamma z\,dz$ where γ is the path consisting of a line segment from 0 to 1 and then a line segment from 1 to $1 + i$.
2. Evaluate $\int_\gamma \sin z\,dz$ along the line segment from 1 to i.
3. Evaluate $\int_\gamma \sin z\,dz$ where γ is the contour $z = e^{it}$ $(0 \le t \le 2\pi)$.

Solutions.

1. We first note that $\frac{z^2}{2}$ is a primitive for z on the whole of $\mathbb{C}$, and γ is a path. Hence
$$\int_\gamma z\,dz = \left[\frac{z^2}{2}\right]_\gamma = \frac{(1+i)^2}{2} - \frac{0^2}{2} = i.$$
2. We note that $-\cos z$ is a primitive for $\sin z$ on $\mathbb{C}$. Hence
$$\int_\gamma \sin z\,dz = [-\cos z]_\gamma = -\cos i - (-\cos 1) = \cos 1 - \cosh 1$$
(using $\cos iz = \cosh z$).
3. Again using the fact that $-\cos z$ is a primitive of $\sin z$ on $\mathbb{C}$ and γ is a contour, we see that $\int_\gamma \sin z\,dz = [-\cos z]_\gamma = 0.$

   Note that in this case, the shape of the contour doesn't matter.
   In fact, $\int_\gamma \sin z\,dz = 0$ for all contours in $\mathbb{C}$. This works for $\int_\gamma f(z)\,dz$ whenever f has a primitive on $\mathbb{C}$ and γ is a contour.

7.3 ML estimates for $\int_\gamma f$

Lemma 7.2 *Suppose that $g : [a,b] \to \mathbb{C}$ is continuous (so g is a function of a real variable, but $g(t)$ is complex- valued). Then $|\int_a^b g(t)dt| \le \int_a^b |g(t)|dt$.* $\square$

Proof. If $\int_a^b g(t)\,dt = 0$, then the result is trivially true.

If $\int_a^b g(t)\,dt \neq 0$, it is a non-zero complex number and so it can be expressed in modulus-argument form as

$$\int_a^b g(t)\,dt = re^{i\phi}\,,$$

where $r = \left|\int_a^b g(t)\,dt\right|$ and $-\pi < \phi \le \pi$. Thus

$$\begin{aligned}
\left|\int_a^b g(t)\,dt\right| &= r = e^{-i\phi}\int_a^b g(t)\,dt \quad \text{(real valued)} \\
&= \int_a^b e^{-i\phi} g(t)\,dt \quad \text{(real valued)} \\
&= \mathrm{Re}\left(\int_a^b e^{-i\phi} g(t)\,dt\right) \\
&= \int_a^b \mathrm{Re}\,(e^{-i\phi} g(t))\,dt \\
&\le \int_a^b |\,\mathrm{Re}\,(e^{-i\phi} g(t))|\,dt \\
&\le \int_a^b |\,e^{-i\phi} g(t)|\,dt \\
&\quad \text{(using rules for real-valued functions)} \\
&= \int_a^b |\,e^{-i\phi}|\,|\,g(t)|\,dt \\
&= \int_a^b |\,g(t)|\,dt
\end{aligned}$$

(as $|e^{-i\phi}| = 1$) and the proof is complete. $\square$

Theorem 7.3 *Suppose that f is continuous on a path γ. Let γ have length L and suppose that $|f(z)| \le M$ on γ. Then $|\int_\gamma f| \le ML$.* $\square$

Proof. Using the Lemma, we see that,

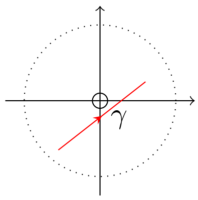
$$\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(z(t))z'(t)\,dt\right| \le \int_a^b |f(z(t))||z'(t)|\,dt \le \int_a^b M|z'(t)|\,dt = ML\,,$$

since $L = \int_a^b |z'(t)|\,dt$. $\square$

7.4 Example

Let γ be a line segment lying within $D = \{z \in \mathbb{C} : |z| < 1\}$. Estimate $\displaystyle\int_\gamma \left(\frac{\mathrm{Re}\,z + z^2}{3 + \overline{z}}\right)\,dz.$

Solution.



Clearly the length L of any straight line segment, in the disc $\{z \in \mathbb{C} : |z| < 1\}$ can't exceed 2, the diameter of the disc. Now, for all $z \in D$,

$$|\operatorname{Re} z + z^2| \;\leq |\;\operatorname{Re} z| + |z^2| \;\leq\; 1 + 1 \;=\; 2$$

and

$$|\,3 + \overline{z}| \geq |3| - |\overline{z}| \geq 3 - 1 = 2 \qquad \text{giving} \qquad \left|\frac{1}{3+\overline{z}}\right| = \frac{1}{|3+\overline{z}|} \leq \tfrac{1}{2}\,.$$

Hence, for all $z \in D$,

$$\left|\frac{\operatorname{Re} z + z^2}{3 + \overline{z}}\right| \;\leq\; 2 \times \tfrac{1}{2} \;=\; 1$$

and so, in the M, L estimate, we can take $M = 1$. Hence by Theorem 7.3,

$$\left|\int_\gamma \frac{\operatorname{Re} z + z^2}{3 + \overline{z}}\, dz\right| \;\leq\; 2.$$

8 Cauchy's Theorem

8.1 Cauchy's Theorem

Theorem 8.1 (Cauchy's Theorem) *Suppose the function f is analytic on a simply connected region D. Then $\int_\gamma f = 0$ for **all** contours γ in D.* □

Note. Cauchy's Theorem was proved about 1814.

Discussion.

1. We have already seen, in 7.1, that if f has a primitive on D, then $\int_\gamma f = 0$. This hypothesis is very strong and Cauchy's statement is far superior.

   Given an analytic function f it is unusual to be able to find a primitive. Cauchy's Theorem, however, allows us to consider $\int_\gamma f$ without any need to find a primitive. For example, let

$$f(z) = \frac{\sin(z^3)}{1 + z^3}\,.$$

Then the function f is analytic on $\mathbb{C} \setminus \{-1, \exp(\frac{i\pi}{3}), \exp(-\frac{i\pi}{3})\}$ and so it is analytic on the disc $U = \{z \in \mathbb{C} : |z| < 1\}$, which is a simply-connected region. Let γ be any contour in U. Then

$$\int_\gamma \frac{\sin(z^3)}{1+z^3}\, dz = 0$$

by Cauchy's Theorem, and we have found the value of the integral without doing any integration.

2. The hypothesis that D is simply-connected is essential and this condition **must always be checked before applying Cauchy's Theorem** as the next example illustrates.

   We know that, if γ is the contour $z = Re^{it} \quad (0 \leq t \leq 2\pi)$, then $\int_\gamma \frac{dz}{z} = 2\pi i$. But $\frac{1}{z}$ is analytic on $C^* = \mathbb{C} \setminus \{0\}$ and γ is a contour in $C^* = \mathbb{C} \setminus \{0\}$. The region C^*, however, is not simply-connected and Cauchy's Theorem could not have been used.

3. Green's Theorem gives some evidence for believing Cauchy's Theorem. Suppose that the function f is analytic in a simply-connected region Ω containing the contour γ. Use the standard notation $z = x + iy$ and $f = u + iv$. Then u, v satisfy the Cauchy-Riemann equations

$$u_x = v_y \qquad v_x = -u_y\,,$$

   on Ω. Let Δ be the region inside the contour γ. Then, using Green's Theorem,

$$\begin{aligned}\int_\gamma f(z)dz &= \int_\gamma (u+iv)(dx+idy)\\ &= \int_\gamma (udx - vdy) + i\int_\gamma (udy + vdx)\\ &= \int\int_\Delta (-v_x - u_y)dxdy + i\int\int_\Delta (u_x - v_y)dxdy\end{aligned}$$

   and both integrals vanish using the Cauchy-Riemann equations. This is not sufficient to prove Cauchy's Theorem, as we assumed that u and v have continuous first partial derivatives in order to apply Green's Theorem.

8.2 Examples

Decide whether Cauchy's Theorem can be used to help evaluate the following integrals and evaluate those you can.

1. $\displaystyle\int_\gamma \frac{dz}{z^2+4}$, where γ is a contour lying in $U = \{z \in \mathbb{C} : |z| < 1\}$;
2. $\int_\gamma \operatorname{Re} z \, dz$ with γ as in question 1. above;
3. $\displaystyle\int_\gamma \frac{z^2+3}{(z+1)\,e^z}\, dz$ with γ as in question 1. above;
4. $\displaystyle\int_\gamma \frac{z^2+3}{(z+1)\,e^z}\, dz$ with γ given by $z(t) = 2e^{it}$ $(0 \leq t \leq 2\pi)$.

Solutions.

1. As $\dfrac{1}{z^2+4}$ is analytic on $\mathbb{C} \setminus \{\pm 2i\}$, it is analytic on the disc U, which is a simply-connected region containing the contour γ. By Cauchy's Theorem
$$\int_\gamma \frac{dz}{z^2+4} = 0\,.$$

2. Here Cauchy's Theorem is irrelevant as $\operatorname{Re} z$ is not analytic on any region. The value of the integral will depend on γ.

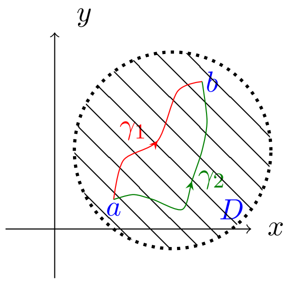
3. As $\dfrac{z^2+3}{(z+1)\,e^z}$ is analytic on $\mathbb{C} \setminus \{-1\}$, it is analytic on the disc U, which is a simply-connected region containing the contour γ. By Cauchy's Theorem
$$\int_\gamma \frac{z^2+3}{(z+1)\,e^z}\, dz = 0\,.$$

4. Now γ is a contour, and $\dfrac{z^2+3}{(z+1)\,e^z}$ is analytic on $\mathbb{C} \setminus \{-1\}$. But -1 belongs to the interior of γ and, therefore, there is no simply connected region containing γ in which the integrand is analytic. Cauchy's Theorem is not applicable if the contour contains any "bad points" of the function. We will develop a method to evaluate this sort of integral later, which avoid real integration techniques.

8.3 Independence of path

Theorem 8.2 *Let f be analytic on a simply-connected region D and let γ_1 and γ_2 be any two paths in D from a to b. Then*
$$\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz\,. \qquad \square$$

Proof. The function f is analytic on the simply-connected region D and $\gamma_1 - \gamma_2$ is a contour in D.



By Cauchy's Theorem

$$\begin{aligned} \int_{\gamma_1 - \gamma_2} f(z)\,dz &= 0 \\ \text{and so} \quad \int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz &= 0\,, \end{aligned}$$

i.e.

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz.$$

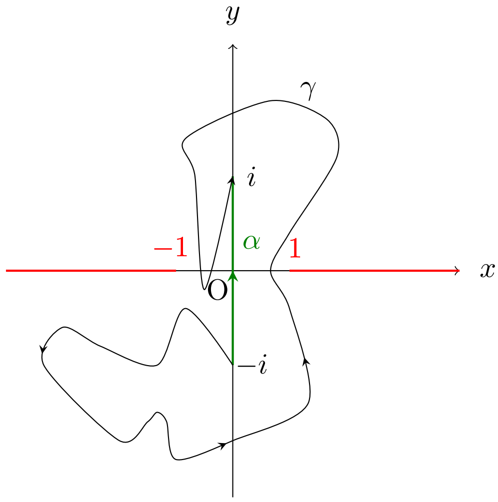
$\square$

8.4 Example

Let γ be any path from $-i$ to i which crosses $\mathbb{R}$ only between -1 and 1. Evaluate $\displaystyle\int_\gamma \frac{dz}{1-z^2}$.

Solution. Let λ be the straight line segment from $-i$ to i and let

$$f(z) = \frac{1}{1-z^2}\ .$$



Then the function f is analytic $\mathbb{C} \setminus \{\pm 1\}$ and so f is analytic on the cut plane $C^* = \mathbb{C} \setminus \{x \in \mathbb{R} : |x| \geq 1\}$ which is a simply-connected region which contains γ and λ. By Cauchy's theorem $\int_\gamma f(z)\,dz = \int_\lambda f(z)\,dz$. Thus

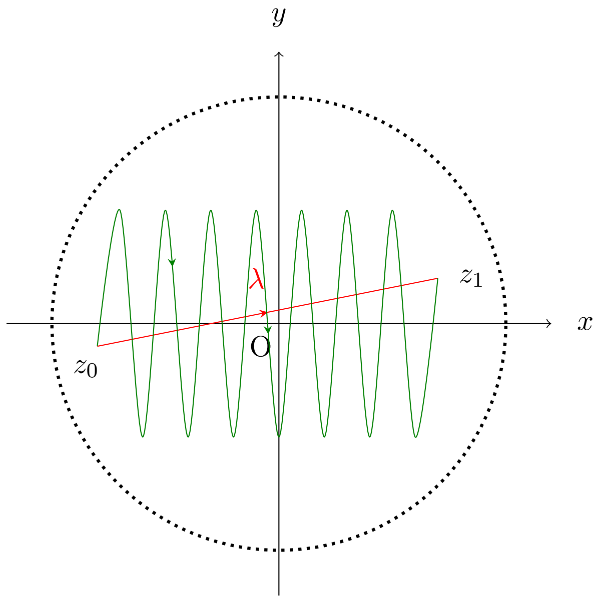
$$\int_\gamma \frac{dz}{1-z^2} = \int_\lambda \frac{dz}{1-z^2} = \int_{-1}^{1} \frac{i\,dy}{1+y^2} = i\left[\tan^{-1} y\right]_{-1}^{1} = i\frac{\pi}{2}\ .$$

8.5 Application to ML-estimates

Example. Let α be any path in $D = \{z \in \mathbb{C} : |z| < 2\}$. Find B so that

$$\left| \int_\alpha \frac{\sinh z}{9+e^z}\,dz \right| \leq B\ .$$

Solution. Suppose that the path α has initial and final points z_0 and z_1 respectively. Let λ be the straight line segment from z_0 to z_1. Then $\lambda \subset D$.



Now $\sinh z$ is analytic on $\mathbb{C}$ and for all $z \in D$,

$$|\, e^z| \;=\; e^{\operatorname{Re} z} \;<\; e^2 \;<\; 9$$

as e^x is an increasing function on $\mathbb{R}$. Hence $9 + e^z$ is analytic and non-zero on D. Thus $\dfrac{1}{9+e^z}$ is analytic on D and, therefore, $\dfrac{\sinh z}{9+e^z}$ is analytic on the disc D, which is a simply connected region.

Hence by theorem 8.2,

$$\left| \int_\alpha \frac{\sinh z}{9+e^z}\, dz \right| \;=\; \left| \int_\lambda \frac{\sinh z}{9+e^z}\, dz \right| \;\leq\; ML\,,$$

where $\left|\dfrac{\sinh z}{9+e^z}\right| \;\leq\; M$ on λ and the length of λ is $|z_1 - z_0|$ which is less than 4, so $L < 4$ in the estimate.

To compute M, we note that for all $z \in D$,

$$|\, 9 + e^z| \;\;\geq\;\; |\, 9| - |\, e^z| \geq 9 - e^2$$

and

$$\begin{aligned} |\sinh z| &= \left|\tfrac{1}{2}(e^z - e^{-z})\right| \\ &\le \tfrac{1}{2}(|e^z| + |e^{-z}|) \\ &= \tfrac{1}{2}(e^{\operatorname{Re} z} + e^{\operatorname{Re}(-z)}) = \tfrac{1}{2}(e^x + e^{-x}) \\ &= \cosh x \\ &\le \cosh 2. \end{aligned}$$

So we can take $M = \dfrac{\cosh 2}{9 - e^2}$, and so

$$B = ML = \frac{4\cosh 2}{9 - e^2}.$$

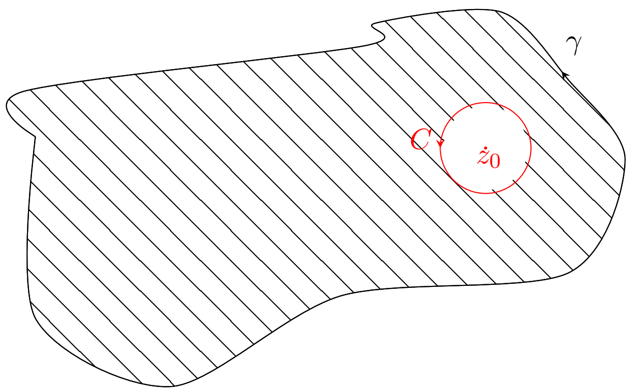
9 Cauchy's Integral Formula (CIF)

9.1 Deforming contours

Theorem 9.1 *Let γ be a simple (see section 3.1) contour described in the positive direction. Let z_0 be a point inside γ, and let $C : z = z_0 + re^{it}$ $(0 \le t \le 2\pi)$ where r is small enough for C to lie inside γ. Suppose f is analytic on a region D which contains γ and C and all points in between (so D need not be simply-connected). Then*

$$\int_\gamma f = \int_C f. \qquad \square$$

Proof. Draw two non-intersecting lines l_1 and l_2 joining C and γ. Suppose l_1 is the line segment from the point a_1 on γ to the point b_1 on C, and that l_2 goes from the point a_2 on γ to b_2 on C. Then γ is broken up into two parts, γ_1 from a_1 to a_2 and γ_2 from a_2 to a_1. Likewise, C is broken up into two parts, C_1 from b_1 to b_2 and C_2 from b_2 to b_1.



Then the contour

$$\gamma_1 + l_2 - C_1 - l_1$$

is a contour going from a_1 to itself, contained in a simply connected region D_1 in which the function f is analytic. Similarly, the contour

$$\gamma_2 + l_1 - C_2 - l_2$$

is a contour going from a_2 to itself, contained in a simply connected region D_2 in which the function f is analytic. We apply Cauchy's Theorem to the simply connected regions D_1 and D_2 to find that

$$\int_{\gamma_1} f + \int_{l_2} f - \int_{C_1} f - \int_{l_1} f = 0$$

and

$$\int_{\gamma_2} f + \int_{l_1} f - \int_{C_2} f - \int_{l_2} f = 0.$$

Adding these and using the fact that $\gamma = \gamma_1 + \gamma_2$ and $C = C_1 + C_2$. We get

$$\int_{\gamma} f - \int_{C} f = 0$$

and the result follows. $\square$

So Theorem 9.1 says that you can deform one contour into another one without changing the value of the integral provided the integrand is analytic between the two contours.

9.2 Cauchy's integral formula

Theorem 9.2 (Cauchy's integral formula) *Let γ be a simple contour described in the positive direction. Let w lie inside γ. Suppose that the function f is analytic on a simply-connected region D containing γ and its interior. Then*

$$f(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - w}\, dz. \qquad \square$$

Proof. Let C be the circular contour given by $z = w + re^{it}$ $(0 \le t \le 2\pi)$, where $r > 0$ is small enough for C to lie inside γ. Then $\frac{f(z)}{z-w}$ is analytic on D except at w, and the hypothesis of Theorem 9.1 holds for the region $D \setminus \{w\}$. Hence

$$\begin{aligned}
\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - w} dz &= \frac{1}{2\pi i} \int_{C} \frac{f(z)}{z - w}\, dz \\
&= \frac{1}{2\pi i} \int_{C} \frac{f(z) - f(w)}{z - w}\, dz + \frac{1}{2\pi i} \int_{C} \frac{f(w)}{z - w}\, dz \\
&= I + \frac{f(w)}{2\pi i} \int_0^{2\pi} \frac{rie^{it}}{re^{it}}\, dt \\
&= I + f(w)
\end{aligned}$$

So

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - w}\, dz - f(w) = I$$

and the left hand side is independent of r. **Hence the value of I is independent of r.** We now prove that $I = 0$ by using the ML-estimates .

Let $M(r)$ be the maximum value of $|f(z) - f(w)|$ for z on C. Then $M(r) \to 0$ as $r \to 0$ because the function f is continuous at w. Using an ML-estimate, we have

$$|I| = \left| \frac{1}{2\pi i} \int_C \frac{f(z) - f(w)}{z - w}\, dz \right| \leq \frac{1}{2\pi} \frac{M(r)}{r} 2\pi r = M(r) \to 0 \text{ as } r \to 0. \qquad (1)$$

Now the value of I is independent of r and $|I| \leq M(r)$ for all sufficiently small $r > 0$, by equation (1). Since $M(r) \to 0$ as $r \to 0$, we see that the value of $|I|$ is less than every positive real number. Hence $|I| = 0$ and the result now follows. □

Note that we could have formulated it as

$$\int_\gamma \frac{f(z)}{z - w}\, dz = 2\pi i f(w).$$

Observe that the formula gives the value of f at w, a point lying strictly inside γ, in terms of its values on γ.

However, this result is frequently used to evaluate given integrals. To do this we choose a suitable function $f(z)$ so that the given integral takes the form above. Examples of this are given below.

How to use Cauchy's Theorem and Cauchy's Integral Formulae to evaluate integrals of the form $\int_\gamma \frac{f(z)}{z-w}\, dz$.

- First draw a diagram showing the contour γ and the point w.
- **If w is outside γ we use Cauchy's Theorem (CT).**
- **If w is inside γ we use Cauchy's Integral Formula (CIF).**

The shape of γ is not otherwise important.

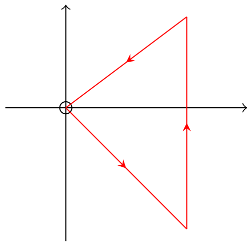
9.3 Examples

Let γ be the simple, positively oriented triangular contour from 0 to $2 - 3i$ to $2 + 2i$ and back to 0. Evaluate

$$(1) \int_\gamma \frac{e^z}{z-1} dz,\ (2) \int_\gamma \frac{e^z}{z+1} dz,\ (3) \int_\gamma \frac{1}{z^2-1} dz,\ (4) \int_\gamma \frac{ze^{z^2}}{(z-1)(2z-1)} dz,\ (5) \int_\gamma \frac{e^{z^2}}{z^2-1} dz.$$

Solutions. We begin by drawing a diagram showing the contour γ and marking the points at which the integrand is not analytic, i.e. we mark the "bad points". We look at where these bad points lie in relation to γ and decide whether Cauchy's Theorem or Cauchy's integral formulae are appropriate or not. The most important point to establish whether the bad points are inside the contour or outside it. The shape of γ is not otherwise

important.
In these examples γ is the triangular contour from 0 to $2-3i$ to $2+2i$ and back to 0.



We note that γ is a simple contour described in the positive direction. This will be used in the solution to questions 1,3,4,5.

1. Let $f(z) = e^z$. Then f is analytic in $\mathbb{C}$ which is a simply-connected region containing γ. The point 1 is inside γ. By Cauchy's integral formula,
$$\int_\gamma \frac{e^z}{z-1} dz = \int_\gamma \frac{f(z)}{z-1}\, dz = 2\pi i f(1) = 2\pi i e.$$

2. Let $g(z) = \frac{e^z}{z+1}$. Here, the 'bad' point -1 lies outside γ, and so we use Cauchy's Theorem. Now the function g is analytic in the half-plane $H = \{z \in \mathbb{C} : \text{Re } z > -1\}$, which is a simply-connected region containing the contour γ. By Cauchy's Theorem,
$$\int_\gamma \frac{e^z}{z+1}\, dz = \int_\gamma g(z)\, dz = 0.$$

3. We could use partial fractions to see that,
$$\int_\gamma \frac{1}{z^2-1}\, dz = \frac{1}{2}\left[\int_\gamma \frac{1}{z-1}\, dz - \int_\gamma \frac{1}{z+1}\, dz\right].$$
Cauchy's integral formula could then be used to evaluate the first integral on the RHS and Cauchy's Theorem gives 0 as the value of the second integral. However, there is a way of avoiding partial fractions altogether in this case.

   Let $h(z) = \frac{1}{z+1}$. Then h is analytic in the half-plane H used in question 2 above. Now the point 1 is inside γ. By Cauchy's integral formula
$$\int_\gamma \frac{1}{z^2-1}\, dz = \int_\gamma \frac{1}{(z+1)(z-1)}\, dz = \int_\gamma \frac{h(z)}{z-1}\, dz = 2\pi i h(1) = \pi i.$$

   In general, it is a good idea to avoid the use of partial fractions, if possible. The idea is to take factors in the denominator up into the function in the numerator if they are non-zero on the simply connected region. This shortens the work <u>and</u> avoids careless mistakes in the partial fractions!!!

4. We note that $\frac{ze^{z^2}}{(z-1)(2z-1)}$ is analytic in $\mathbb{C}\setminus\{\frac{1}{2}, 1\}$. The bad points, $\frac{1}{2}$ and 1 both lie inside γ, so we must use partial fractions. There is no way of avoiding them this time! We have

$$\frac{1}{(z-1)(2z-1)} = \frac{1}{2\,(z-1)\left(z-\frac{1}{2}\right)} = \frac{1}{(z-1)} - \frac{1}{\left(z-\frac{1}{2}\right)}. \tag{1}$$

Let $k(z) = z\,e^{z^2}$. Then k is analytic in $\mathbb{C}$, which is a simply-connected region containing γ. Since the points $\frac{1}{2}$, 1 lie inside γ, Cauchy's integral formula gives,

$$\begin{aligned}\int_\gamma \frac{ze^{z^2}}{(z-1)(2z-1)}\,dz &= \int_\gamma \frac{k(z)}{2(z-1)(z-\frac{1}{2})}\,dz &&= \int_\gamma \frac{k(z)}{z-1}\,dz - \int_\gamma \frac{k(z)}{z-\frac{1}{2}}\,dz \\ &= 2\pi i k(1) - 2\pi i k\left(\tfrac{1}{2}\right) &&= 2\pi i\left(e - \tfrac{1}{2}\,e^{\frac{1}{4}}\right),\end{aligned}$$

using (1).

5. This time we can avoid the use of partial fractions. Let $m(z) = \frac{e^{z^2}}{z+1}$ and let H be the half-plane used in the solution of question 3, i.e. $H = \{z \in \mathbb{C} : \mathrm{Re}\, z > -1\}$. then m is analytic in H, which is a simply-connected region containing γ. Since the point 1 lies in side γ, Cauchy's integral formula gives

$$\int_\gamma \frac{e^{z^2}}{z^2-1}\,dz = \int_\gamma \frac{e^{z^2}}{(z+1)(z-1)}dz = \int_\gamma \frac{m(z)}{z-1}\,dz = 2\pi i m(1) = \pi i e\,.$$

9.4 Cauchy's integral formula for the derivatives

Theorem 9.3 *[Cauchy's integral formula for the derivatives] Let γ be a simple contour described in the positive direction. Let w be any point inside γ. Suppose that the function f is an analytic on a simply-connected region D containing γ and its interior. Then, for all $n \in \mathbb{N}$,*

$$f^{(n)}(w) = \frac{n!}{2i\pi}\int_\gamma \frac{f(z)}{(z-w)^{n+1}}dz,$$

or equivalently,

$$\int_\gamma \frac{f(z)}{(z-w)^{n+1}}dz = \frac{2i\pi}{n!}f^{(n)}(w). \qquad \square$$

Proof. CIF says that

$$f(w) = \frac{1}{2i\pi}\int_\gamma \frac{f(z)}{z-w}dz.$$

Differentiate with respect to w (which we denote $'$):

$$\begin{aligned} f'(w) &= \frac{1}{2i\pi}\left(\int_\gamma \frac{f(z)}{z-w}dz\right)' \\ &= \frac{1}{2i\pi}\int_\gamma \left(\frac{f(z)}{z-w}\right)' dz \\ &= \frac{1}{2i\pi}\int_\gamma \frac{f(z)}{(z-w)^2}dz \end{aligned}$$

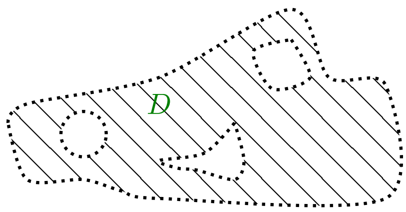
where we assume that the integration and differentiation may be switched. Differentiating again as many times as necessary, to get the result. Note that

$$\left(\frac{1}{z-w}\right)^{(n)} = \frac{n!}{(z-w)^{n+1}}.$$

Thus we get Cauchy's Integral Formula for the nth derivative, which will be denoted by $(CIF)^{(n)}$ when we wish to emphasize which derivative is involved. $\square$

Corollary 1. If f is analytic on a region D, then f has derivatives of all orders at each point of D and each derivative is analytic on D.

Proof. Take any point $w \in D$. As D is a region, it is open, so we can find a disc $\Delta = \{z \in \mathbb{C} : |z-w| < R\}$ about w contained in D. Let γ be the circular contour $z = w + re^{it} \quad (0 \le t \le 2\pi)$, where $0 < r < R$.



Then f is analytic in the simply-connected region Δ containing the simple contour γ, which is described in the positive direction. Since the point w is inside γ, Cauchy's integral formula for the nth derivative implies that $f^{(n)}(w)$ exists for all $n \in \mathbb{N}$. As $w \in D$ is arbitrary we see that $f^{(n)}$ exists on D.

Since $f^{(n+1)}$ exists on D, it follows that $f^{(n)}$ is differentiable on D and so $f^{(n)}$ is analytic on the region D. This is true for all $n \in \mathbb{N}$. $\square$

Corollary 2 (Theorem 5.9). Suppose $f = u + iv$ is analytic on a region D. Then u and v are harmonic on D.

Proof. (Proper) By Corollary 1, f' is analytic on D. Write

$$f' = u_x - iu_y = U + iV \qquad \text{where } U = u_x\,, \quad V = -u_y\,.$$

Then U and V satisfy the first Cauchy-Riemann equation $U_x = V_y$, i.e. $(u_x)_x = (-u_y)_y$, so $u_{xx} + u_{yy} = 0$ on D.

As $f' = v_y + iv_x$ satisfies the second Cauchy-Riemann equation, we see that $(v_y)_y = -(v_x)_x$, so $v_{xx} + v_{yy} = 0$ on D. $\square$

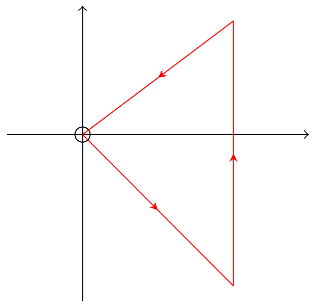
9.5 Examples

Let γ be the simple, positively oriented triangular contour from 0 to $2-3i$ to $2+2i$ and back to 0. Evaluate

(1) $\displaystyle\int_\gamma \frac{\sin z}{(z-1)^{2n+1}}\, dz$, (2) $\displaystyle\int_\gamma \frac{\sin z}{(z+1)^{2n+1}}\, dz$, (3) $\displaystyle\int_\gamma \frac{\cos(z^2)}{(2z-1)^2}\, dz$,

(4) $\displaystyle\int_\gamma \frac{e^z}{(z-1)^{10}}\,dz$, (5) $\displaystyle\int_\gamma \frac{e^z}{(z^2-1)^2}\,dz$.

Solutions. In these examples γ is the triangular contour from 0 to $2-3i$ to $2+2i$ and back to 0.



We note that γ is a simple contour described in the positive direction. This will be used in the solution to questions 1,3,4,5.

1. Let $f(z) = \sin z$. Then f is analytic in $\mathbb{C}$, which is a simply-connected region containing the contour γ. Since the point 1 lies inside γ, we use $(CIF)^{(2n)}$ with $w = 1$. Now the $(2n)^{\text{th}}$ derivative of $\sin z$ is $(-1)^n \sin z$ and so
$$\int_\gamma \frac{\sin z}{(z-1)^{2n+1}}\,dz = \int_\gamma \frac{f(z)}{(z-1)^{2n+1}}\,dz = \frac{2\pi i}{(2n)!}\,f^{(2n)}(1) = \frac{2\pi i}{(2n)!}(-1)^n \sin 1.$$

2. The bad point -1 lies outside γ, and $\frac{\sin z}{(z+1)^{2n+1}}$ is analytic on the half-plane $H = \{z \in \mathbb{C} : \mathrm{Re}\ z > -1\}$ which is a simply connected region containing the contour γ. Thus, by Cauchy's Theorem
$$\int_\gamma \frac{\sin z}{(z+1)^{2n+1}}\,dz = 0.$$

3. Let $g(z) = \cos(z^2)$. Then g is analytic in $\mathbb{C}$, which is a simply-connected region containing γ. The point $\frac{1}{2}$ lies inside γ. Using Cauchy's integral formula for the first order derivative, we see that,
$$\begin{aligned}\int_\gamma \frac{\cos(z^2)}{(2z-1)^2}\,dz &= \frac{1}{4}\int_\gamma \frac{\cos(z^2)}{(z-\frac{1}{2})^2}\,dz &&= \frac{1}{4}\int_\gamma \frac{g(z)}{(z-\frac{1}{2})^2}\,dz \\ &= \tfrac{1}{4}\,\tfrac{2\pi i}{1!}\,g'\left(\tfrac{1}{2}\right) &&= \tfrac{\pi i}{2}\left[-2z\sin(z^2)\right]_{z=\frac{1}{2}} \quad = \quad -\tfrac{\pi i}{2}\sin\tfrac{1}{4}.\end{aligned}$$

4. Let $h(z) = e^z$. Then h is analytic in $\mathbb{C}$, which is a simply- connected region containing γ. Since the point 1 is inside γ, Cauchy's integral formula for the 9th derivative gives,
$$\int_\gamma \frac{e^z}{(z-1)^{10}}\,dz = \int_\gamma \frac{h(z)}{(z-1)^{10}}\,dz = \frac{2\pi i}{9!}\,h^{(9)}(1) = \frac{2\pi i e}{9!}.$$

5. Let $k(z) = \frac{e^z}{(z+1)^2}$. Then k is analytic on the half-plane $H = \{z \in \mathbb{C} : \text{Re } z > -1\}$ which is a simply connected region containing the contour γ. Since the point 1 is inside γ, Cauchy's integral formula for the first derivative gives,

$$\begin{aligned}
\int_\gamma \frac{e^z}{(z^2-1)^2}\,dz &= \int_\gamma \frac{e^z}{(z-1)^2(z+1)^2}\,dz &= \int_\gamma \frac{k(z)}{(z-1)^2}\,dz \\
&= \frac{2\pi i}{1!}\,k'(1) &= 2\pi i\left(\frac{e}{4}-\frac{2e}{8}\right) = 0,
\end{aligned}$$

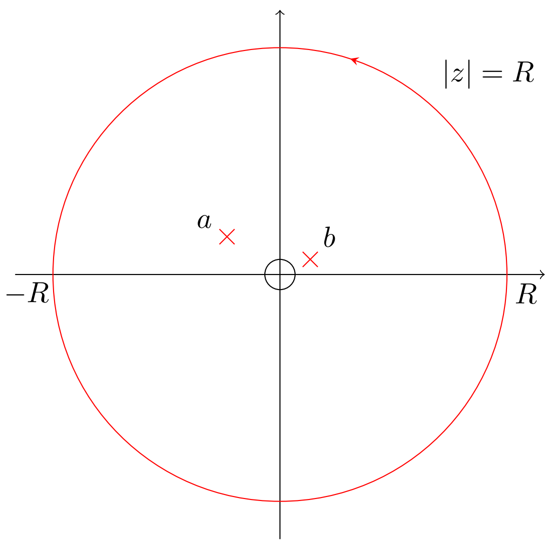
using the fact that $k'(z) = \frac{e^z}{(z+1)^2} - \frac{2e^z}{(z+1)^3}$.

9.6 Liouville's Theorem

Using Cauchy's Integral Formulae we can prove Liouville's Theorem

Theorem 9.4 (Liouville's Theorem) *A function which is analytic and bounded in the complex plane is a constant.*

Proof. Suppose that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Choose any two distinct points a and b in $\mathbb{C}$ and choose R so that $R \geq 2\max\{|a|, |b|\}$ and let γ be the contour given by $z = Re^{it}$ $(0 \leq t \leq 2\pi)$.



Then using Cauchy's Formula

$$\begin{aligned}
f(b) - f(a) &= \frac{1}{2\pi i}\int_\gamma f(w)\left(\frac{1}{w-b} - \frac{1}{w-a}\right)dw \quad (1)\\
&= \frac{b-a}{2\pi i}\int_\gamma \frac{f(w)}{(w-b)(w-a)}\,dw\,,
\end{aligned}$$

Now for all w on γ, $|w-a| > \frac{1}{2}R$ and $|w-b| > \frac{1}{2}R$ and so $|(w-a)(w-b)| < \frac{1}{4}R^2$ on γ. It follows that

$$\begin{aligned} |f(b)-f(a)| &= \left|\frac{b-a}{2\pi i}\right| \left| \int_\gamma \frac{f(w)}{(w-b)(w-a)}\, dw \right| \\ &\leq \frac{|b-a|}{2\pi} . \frac{2\pi R 4M}{R^2} = \frac{4|b-a|M}{R} \end{aligned}$$

The righthand side of this inequality can be made arbitarily small by making R sufficiently large and hence $|f(b) - f(a)|$ is less than every positive real number, however small, i.e. $f(b) = f(a)$. This is true for all points a and b in $\mathbb{C}$. Hence f is constant in $\mathbb{C}$. □

9.7 Corollary

Using Liouville's Theorem, we can prove the Fundamental Theorem of Algebra

Corollary 9.5 *(Fundamental Theorem of Algebra) Let p(z) be a non-constant polynomial with complex coefficients, then there is a point* $w \in \mathbb{C}$ *such that* $p(w) = 0$.

Proof. We use a contradiction argument.
Suppose that the non-constant polynomial p is non-zero in $\mathbb{C}$ and let $g(z) = \dfrac{1}{p(z)}$ for all $z \in \mathbb{C}$, then g is analytic in $\mathbb{C}$. Now p is a non-constant polynomial and so $|p(z)| \to \infty$ as $|z| \to \infty$. Hence $|g(z)| = \left|\frac{1}{p(z)}\right| \to 0$ as $|z| \to \infty$ and so there exists $R > 0$ such that $|g(z)| < 1$ for all z such that $|z| > R$. **(1)**

Now $\overline{D} = \{z \in \mathbb{C} : |z| \leq R\}$ is a closed bounded set in $\mathbb{C}$ and so the analytic function g is bounded on $\overline{D}$ and there exists M such that $|g(z)| \leq M$ for all $z \in \overline{D}$ **(2)**.

From (1) and (2) we see that g is analytic and bounded on $\mathbb{C}$ and so by Liouville's Theorem g is constant, giving $p = \frac{1}{g}$ is also a constant, which is a contradiction.

Thus $p(z)$ has a zero w in $\mathbb{C}$.
From this we can deduce, using Mathematical Induction, that a polynomial of degree n with complex coefficients has precisely n complex roots, where multiple roots are counted according to their multiplicity.

Note. If p has degree n, where $n > 1$, then there is a complex number w such that $p(w) = 0$ and so we can write $p(z) = (z - w)q(z)$ where q is a polynomial of degree $(n-1)$ and the Induction proof is obvious. □

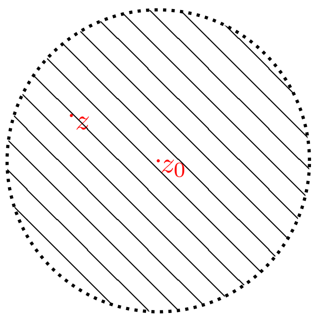
10 Taylor's Theorem

10.1 Taylor's Theorem

Theorem 10.1 (Taylor's Theorem) *Let the function f be analytic in the disc $\Delta = \{z \in \mathbb{C} : |z - z_0| < r\}$, where $r > 0$. Then $f(z)$ has a Taylor expansion about z_0 valid on Δ. For all $z \in \Delta$,*

$$f(z) = f(z_0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

Proof. Consider $z \in \Delta$, and choose $\rho \in \Delta$ such that $|z - z_0| < \rho < r$. Let γ_ρ be the circular contour given by $w = z_0 + \rho e^{it}$ $(0 \leq t \leq 2\pi)$. Then Δ is a simply connected region containing γ_ρ.



Δ is the shaded open disc in the above diagram.

Using Cauchy's Integral Formula gives

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_\rho} \frac{f(w)}{w - z}\, dw\,. \tag{1}$$

We expand $\frac{1}{w-z}$ in powers of $(z - z_0)$ and $(w - z_0)$. Write

$$\begin{aligned}
\frac{1}{w-z} &= \frac{1}{(w-z_0)-(z-z_0)} = \frac{1}{(w-z_0)}\left(1 - \frac{z-z_0}{w-z_0}\right)^{-1} \\
&= \frac{1}{(w-z_0)}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n = \sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(w-z_0)^{n+1}}\,.
\end{aligned}$$

(Note that $|\frac{z-z_0}{w-z_0}| < 1$ for $w \in \gamma_\rho$.)

Using (1) gives

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\gamma_\rho}\frac{f(w)}{w-z}dw \\ &= \frac{1}{2\pi i}\int_{\gamma_\rho}\sum_{n=0}^{\infty}\frac{f(w)(z-z_0)^n}{(w-z_0)^{n+1}}dw \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\gamma_\rho}\frac{f(w)}{(w-z_0)^{n+1}}dw\right)(z-z_0)^n \\ &= \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \end{aligned}$$

using $CIF^{(n)}$ after switching orders of integration and summation. $\square$

10.2 Example

Find the Taylor series expansion of $\frac{1}{1-z^2}$ about $z=0$. Where is this expansion valid?

Solution. We use the familiar relation

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots \qquad (|t| < 1) \tag{*}$$

with t replaced by z^2. This gives

$$\frac{1}{1-z^2} = 1 + z^2 + z^4 + z^6 + \cdots = \sum_{n=0}^{\infty} z^{2n}$$

for $|z| < 1$. This is the Taylor series for $\frac{1}{1-z^2}$ valid in the disc $D = \{z \in \mathbb{C} : |z| < 1\}$.

This method was much easier than differentiating $\frac{1}{1-z^2}$ and evaluating the derivatives at the origin. But, how do we know that this is the Taylor series since we did not differentiate to find the Taylor coefficients?

We can show that a function analytic in a disc $\Delta = \{z \in \mathbb{C} : |z - z_0| < r\} \quad (r > 0)$, has one and only one series expansion in powers of $(z - z_0)$ valid in Δ, namely the Taylor series. This is called the uniqueness theorem for Taylor series. In view of this result you can use the easiest method, in any given problem, to obtain a valid series expansion and you can be sure that this expansion is, indeed, the Taylor series.

In practice, we will usually obtain the Taylor series by manipulating known expansions and avoid differentiating.

Theorem 10.2 *(Uniqueness Theorem for Taylor series expansions) Suppose that the function f is analytic in the disc $\Delta = \{z \in \mathbb{C} : |z - z_0| < r\}$, where $r > 0$. Suppose also that*

$$f(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)^2 + \cdots = \sum_{n=0}^{\infty} b_n (z - z_0)^n \qquad (I)$$

on Δ. Then

$$b_n = \frac{f^{(n)}(z_0)}{n!} \qquad (n = 0, 1, 2, 3, \cdots)$$

i.e. $\sum_{n=0}^{\infty} b_n (z - z_0)^n$ is the Taylor series for $f(z)$ on Δ.

Proof. Since

$$f(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n\,, \quad f(z_0) = b_0\,.$$

Now a power series can be differentiated term by term, any number of times, in its disc of convergence. Differentiate n times, in the disc of convergence, and put $z = z_0$, to obtain

$$f^{(n)}(z_0) = n!\, b_n \quad \text{i.e.} \quad b_n = \frac{f^{(n)}(z_0)}{n!}\,.$$

Thus $\sum_{n=0}^{\infty} b_n (z - z_0)^n$ is the Taylor series. $\square$

10.3 Example

1. Find the Taylor series of $\dfrac{1}{z-1}$ about $z = 2$.

<u>Note</u> **When finding Taylor Series it is frequently useful to use the following relation**

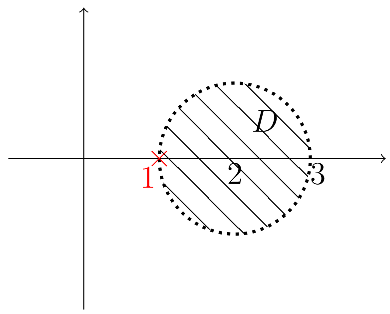
$$\frac{1}{1-t} = 1 + t + t^2 + t^3 = \cdots = \sum_{n=0}^{\infty} t^n \quad (|t| < 1)$$

with a suitable expression in place of t.

In general, if the centre of the expansion is $z_0 \neq 0$, put $w = z - z_0$ and expand in powers of w, then replace w by $(z - z_0)$, to obtain a power series in powers of $(z - z_0)$.

Solution.

1. The function $\frac{1}{z-1}$ is analytic on $\mathbb{C} \setminus \{1\}$ and so it is analytic in the disc $D = \{z \in \mathbb{C} : |z - 2| < 1\}$. This is the largest disc with centre 2 in which f is analytic.



Here, the centre of expansion is 2, and so we put $w = z - 2$ (see note at end of questions). Then, for $|z-2| < 1$ i.e. for $|w| < 1$,

$$\frac{1}{z-1} = \frac{1}{w+1} = \sum_{n=0}^{\infty} (-1)^n w^n$$

using a known expansion, so the Taylor series about $z = 2$, in powers of $(z-2)$, is

$$\frac{1}{z-1} = \sum_{n=0}^{\infty} (-1)^n (z-2)^n ,$$

valid on $\Delta = \{z : |z-2| < 1\}$.

10.4 Zeros

Definition 10.3 *Suppose that f is analytic on a region D and $f(w) = 0$ for some $w \in D$. Then w is a zero of f.*

If the function f has a zero at the point w, then $f(w) = 0$ and the Taylor expansion of $f(z)$ about w is of the form

$$\begin{aligned} f(z) &= f(w) + f'(w)(z-w) + \tfrac{f''(w)}{2!}(z-w)^2 + \cdots \\ &= f'(w)(z-w) + \tfrac{f''(w)}{2!}(z-w)^2 + \cdots \\ &= (z-w)g(z), \end{aligned}$$

where

$$g(z) = f'(w) + \frac{f''(w)}{2!}(z-w) + \cdots$$

and so the function g is analytic in some neighbourhood of w.
Of course, we may have $f'(w) = 0$ as well, in which case we can take out a factor $(z-w)^2$ and so on.

Definition 10.4 *Suppose that the function f is analytic in a region D and $w \in D$. If $f(w) = f'(w) = \cdots = f^{(k-1)}(w) = 0$ and $f^{(k)}(w) \neq 0$ (so we can take out a factor $(z-w)^k$), we say that f has a zero of order k at w.*

Thus if the function f has a zero of order k at w, then we can express $f(z)$ in the form $f(z) = (z-w)^k g(z)$ where g is analytic in some neighbourhood of w and $g(w) = \frac{f^{(k)}(w)}{k!} \neq 0$.

If $k = 1$, we say that the zero is a *simple* zero.

Theorem 10.5 *Let k be a positive integer. The function f has a zero of order k at w if and only if*

$$f(z) = (z-w)^k g(z)$$

in some neighbourhood U of w, where the function g is analytic and non-zero on U.

Proof. If the function f has a zero of order k at w, then $f(z)$ has a Taylor expansion about w of the form

$$\begin{aligned} f(z) &= f(w) + \tfrac{f'(w)}{1!}(z-w) + \tfrac{f''(w)}{2!}(z-w)^2 + \cdots \\ &= 0 + 0 + \cdots + 0 + \tfrac{f^{(k)}(w)}{k!}(z-w)^k + \tfrac{f^{(k+1)}(w)}{(k+1)!}(z-w)^{(k+1)} + \cdots \\ &= (z-w)^k \left[\tfrac{f^{(k)}(w)}{k!} + \tfrac{f^{(k+1)}(w)}{(k+1)!}(z-w) + \cdots\right] \\ &= (z-w)^k \, g(z) \ , \end{aligned}$$

where g is analytic and non-zero in some disc U about w, since g is the sum function of a power series about w (with positive radius of convergence) and $g(w) = \frac{f^{(k)}(w)}{k!} \neq 0$.

Conversely, if

$$f(z) = (z-w)^k g(z)$$

in some neighbourhood U of w, where the function g is analytic and non-zero on U, then using Leibnitz Theorem to differentiate the product gives

$$f(w) = 0, \qquad f^{(n)}(w) = 0 \quad (n = 0, ..., (k-1)), \qquad f^{(k)}(w) \neq 0,$$

and, therefore, f has a zero of order k at w. □

Corollary 10.6 *Suppose that*
(i) the function f has a zero of order m at the point w;
(ii) the function g has a zero of order n at the point w.
Write $h(z) = f(z)g(z)$. Then h has a zero of order $m+n$ at w.
i.e. fg has a zero of order $m+n$ at the point w.

10.5 Examples

1. Show that $\sin z$ has a simple zero at $z = 0$.

2. Find the order of the zero of $1 - \cos z$ at $z = 0$.

3. Show that $z \sin(z^2)$ has a zero of order 3 at the origin.

Solutions.

1. Let $f(z) = \sin z$ Then, f is analytic in $\mathbb{C}$ and, for all integers n,

$$f(n\pi) = \sin(n\pi) = 0 \qquad \text{and} \qquad f'(n\pi) = (\cos(n\pi)) = (-1)^n \neq 0.$$

Thus $\sin z$ has a simple zero at $z = 0$. It also has simple zeros at the points $\pm\pi$, $\pm 2\pi$, $\pm 3\pi$, $\cdots$.

2. Let $g(z) = 1 - \cos z$. Then g is analytic in $\mathbb{C}$ and

$$g(0) = 0\,, \qquad g'(0) = \sin 0 = 0\,, \qquad g''(0) = \cos 0 = 1 \neq 0.$$

Hence g has a zero of order 2 at $z = 0$.

3. Let $h(z) = \sin(z^2)$. Then h is analytic on $\mathbb{C}$ and

$$h(0) = 0,\ \ h'(0) = \big(2z\cos(z^2)\big)_{z=0} = 0,\ \ h''(0) = \big(2\cos(z^2) - 4z^2\sin(z^2)\big)_{z=0} = 1 \neq 0.$$

Thus $\sin(z^2)$ has a zero of order 2 at the origin. Since z is analytic in $\mathbb{C}$ and has a zero of order 1 at the origin, we see, from corollary 10.6, that $z\sin(z^2)$ has a zero of order 3 at the origin.

11 Laurent's Theorem

11.1 Doubly infinite series

Let $\alpha \in \mathbb{C}$. In section 6.5 we saw that a power series $\sum_{n=0}^{\infty} a_n(z-\alpha)^n$ has a radius of convergence R_1 , say.

If we consider $\sum_{n=1}^{\infty} \frac{b_n}{(z-\alpha)^n}$, we can view this as a power series in $\frac{1}{z-\alpha}$ and it will converge on a set of the form $\{z : |z-\alpha| > R_2\}$ for some R_2. If we put $a_{-n} = b_n$ for all positive integers n, then

$$\sum_{n=1}^{\infty} \frac{b_n}{(z-\alpha)^n} = \sum_{n=1}^{\infty} a_{-n}(z-\alpha)^{-n} = \sum_{n=-\infty}^{-1} a_n(z-\alpha)^n\,.$$

Suppose $R_1 > R_2$. Then we can expect the doubly infinite series

$$\sum_{n=-\infty}^{\infty} a_n(z-\alpha)^n = \sum_{n=-\infty}^{-1} a_n(z-\alpha)^n + \sum_{n=0}^{\infty} a_n(z-\alpha)^n$$

to have an annulus of convergence $\{z : R_2 < |z-\alpha| < R_1\}$. ($R_2 = 0$ and $R_1 = \infty$ are allowed.) In this course we will only consider the case in which $R_2 = 0$.

Definition 11.1 *A function f which is analytic on the punctured disc $D' = \{z \in \mathbb{C} : 0 < |z-\alpha| < R\}$ but not on $\{z \in \mathbb{C} : |z-\alpha| < R\}$ is said to have an isolated singularity at α.*

So "isolated singularity" is the same as "bad point".

11.2 Examples

1. $\frac{1}{\sin z}$ has singularities at $z = n\pi$ for $n \in \mathbb{Z}$ and is analytic on $\{z : 0 < |z| < \pi\}$ for example.

2. $\frac{1}{(z-1)(z+2)}$ has singularities at 1 and at -2, and is analytic on the punctured discs $\{z : 0 < |z-1| < 3\}$ and $\{z : 0 < |z+2| < 3\}$.

11.3 Laurent's Theorem

Theorem 11.2 (Laurent's Theorem) *Suppose that f has an isolated singularity at α (so f is analytic on some punctured disc $D' = \{z : 0 < |z-\alpha| < R\}$). Then f can be represented on D' by a Laurent series about $z = \alpha$, i.e.,*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-\alpha)^n$$

for $z \in D'$. The Laurent coefficients are given by

$$a_n = \frac{1}{2\pi i} \int_{C_r} \frac{f(w)}{(w-\alpha)^{n+1}} dw,$$

where $C_r : w = \alpha + re^{it}$ $(0 \leq t \leq 2\pi)$ for any $0 < r < R$.

Note. The a_n are independent of r: if $0 < r_1 < r_2 < R$, we know that the function $\frac{f(w)}{(w-\alpha)^{n+1}}$ is analytic on the D' and is therefore analytic between C_{r_1} and C_{r_2}. By Theorem 9.1, we see that

$$\int_{C_{r_1}} \frac{f(w)}{(w-\alpha)^{n+1}}\, dw = \int_{C_{r_2}} \frac{f(w)}{(w-\alpha)^{n+1}}\, dw$$

and so is independent of r.

[Rest of proof omitted.]

We also assume that the Laurent series is uniquely determined (like the Taylor series).

Definition 11.3 *The Laurent coefficient a_{-1} is called the residue of f at α. We write*

$$a_{-1} = \mathrm{Res}\,\{f; \alpha\}. \qquad \square$$

If f has a singularity at α and C_r is a positively oriented circle centred on α of radius r, then

$$\int_{C_r} f(z)dz = 2\pi i\, \mathrm{Res}\,\{f; \alpha\}.$$

We develop means of finding residues independently of this formula and this will enable us to evaluate certain integrals immediately.

11.4 Examples

1. Find the residue of $\frac{\sinh z}{z^4}$ at the origin.

2. Find the Laurent series of $\frac{1}{z(z-1)}$ about $z = 1$ giving an expression for the general term. Where is this expansion valid?

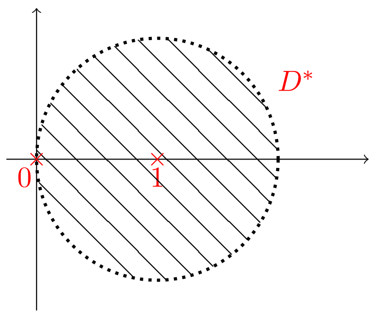
Solutions.

1. Let $f(z) = \frac{\sinh z}{z^4}$. Then f has a singularity at 0 and that it is analytic on $\mathbb{C} \setminus \{0\}$. For $z \neq 0$,

$$f(z) = \frac{\sinh z}{z^4} = \frac{z + \frac{z^3}{6} + \frac{z^5}{120} + \cdots}{z^4} = \frac{1}{z^3} + \frac{\frac{1}{6}}{z} + \frac{z}{120} + \cdots$$

and so $\mathrm{Res}\,\{f : 0\} = \frac{1}{6}$.

2. Let $g(z) = \frac{1}{z(z-1)}$. Then g is analytic in $\mathbb{C} \setminus \{0, 1\}$. It has isolated singularities at the points 0,1. It is analytic on the punctured disc $D^* = \{z \in \mathbb{C} : 0 < |z-1| < 1\}$ about 1. Hence $g(z)$ has a Laurent series expansion, in powers of $z-1$ valid in D^*. Put $w = z-1$. For $0 < |z-1| < 1$, i.e. $0 < |w| < 1$,



$$\begin{aligned} g(z) &= \frac{1}{z(z-1)} &&= \frac{1}{z-1} - \frac{1}{z} = \frac{1}{w} - \frac{1}{1+w} \\ &= \frac{1}{w} - \sum_{n=0}^{\infty} (-1)^n w^n &&= \frac{1}{z-1} - \sum_{n=0}^{\infty} (-1)^n (z-1)^n . \end{aligned}$$

This is the Laurent series expansion for $g(z)$ valid in D^*.

Summary.

If f is analytic at z_0, then $f(z)$ has a Taylor Series expansion $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ valid for z in some disc Δ centred on z_0.

If the function f has an isolated singularity at α, then $f(z)$ has a Laurent Series expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-\alpha)^n$ valid for z in some punctured disc D' centred on α.

11.5 Classification of singularities

Suppose that f has an isolated singularity at α. Then $f(z)$ has a Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-\alpha)^n = \sum_{n=1}^{\infty} \frac{b_n}{(z-\alpha)^n} + \sum_{n=0}^{\infty} a_n (z-\alpha)^n \,,$$

valid in some punctured disc about α, where $b_n = a_{-n}$ for all positive integers n.

A. REMOVABLE SINGULARITY.

Definition 11.4 *If $a_n = 0$ for all $n < 0$, (so that $b_n = 0$ for all positive integers n, i.e. all the negative powers of $(z-\alpha)$ in the Laurent expansion have coefficient 0) then α is called a removable singularity.*

If the function f has a removable singularity at α, then $f(\alpha)$ is undefined (or it has an unsuitable value). If we define $f(\alpha)$ by (or change the value of $f(\alpha)$ so that) $f(\alpha) = a_0$, then we get a function which is defined on a disc centred at α given by a Taylor series, i.e. an analytic function. We have, then, removed the "removable singularity" by defining (or redefining) $f(\alpha)$.
For example, $\frac{\sin z}{z}$ has a singularity at 0 as it is not defined there. But the function

$$f(z) = \begin{cases} \frac{\sin z}{z} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}$$

is analytic on $\mathbb{C}$ and

$$f(z) = 1 - \frac{z^2}{6} + \frac{z^4}{120} - \frac{z^6}{7!} + \cdots$$

is its Taylor series about 0, valid for all $z \in \mathbb{C}$.

B. ISOLATED ESSENTIAL SINGULARITY.

Definition 11.5 *If $a_n \neq 0$ for infinitely many $n < 0$, (so that infinitely many of the coefficients b_n are* ***non-zero*** *i.e. there are infinitely many negative powers of $(z-\alpha)$ with non-zero coefficient in the Laurent expansion) then f has an isolated essential singularity at α.*

For example, $e^{1/z}$ and $\sin\left(\frac{1}{z}\right)$ are both analytic on the punctured disc $D' = \{z \in \mathbb{C} : 0 < |z|\}$, and so $z = 0$ is an isolated singularity of $e^{1/z}$ and of $\sin\left(\frac{1}{z}\right)$. For $z \neq 0$,

$$\begin{aligned} e^{\frac{1}{z}} &= 1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots \,, \\ \sin\left(\tfrac{1}{z}\right) &= \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \cdots \,. \end{aligned}$$

Thus $e^{1/z}$ and $\sin\left(\frac{1}{z}\right)$ both have isolated essential singularities at the origin.
Note The even negative powers of z in the Laurent expansion of $\sin\left(\frac{1}{z}\right)$ about the origin all have coefficient zero. However, there are still an infinite number of negative powers with non-zero coefficient.

C. POLE.

Definition 11.6 *If we can find $k \in \mathbb{N}$ such that $a_{-k} \neq 0$ and $a_n = 0$ for $n < -k$, then f is said to have a pole of order k at α.*

In this case $b_k \neq 0$ and $b_n = 0$ for **all** $n > k$ and the Laurent series contains only a **finite number** of negative powers of $(z-\alpha)$ with **non-zero coefficients**.
The Laurent series looks like

$$\frac{a_{-k}}{(z-\alpha)^k} + \frac{a_{-k+1}}{(z-\alpha)^{k-1}} + \cdots + \frac{a_{-1}}{(z-\alpha)} + a_0 + a_1(z-\alpha) + \cdots .$$

For example, for $z \neq 0$,

$$\frac{\sinh z}{z^4} = \frac{1}{z^3} + \frac{\frac{1}{6}}{z} + \frac{z}{120} + \cdots$$

and so $\frac{\sinh z}{z^4}$ has a pole of order 3 at the origin. Note that 0 is an isolated singularity as the function is analytic on the punctured disc $D' = \{z \in \mathbb{C} : 0 < |z|\}$.

Note that $\sinh z$ has a zero to order 1 at the origin and z^4 has a zero to order 4. The quotient $\dfrac{\sinh z}{z^4}$ therefore has a zero of order -3 in some sense. Let's formalise this idea.

Theorem 11.7 *The function f has a pole of order k at α if and only if $f(z)$ can be expressed in the form*

$$f(z) = \frac{g(z)}{(z-\alpha)^k}$$

in some punctured disc $D' = \{z \in \mathbb{C} : 0 < |z-\alpha| < R\}$, $(R > 0)$ where, g is analytic and non-zero in the disc $D = D' \cup \{\alpha\}$. □

Proof. (i) "Only if" Suppose that f has a pole of order k at $z = \alpha$. Then $f(z)$ has a Laurent expansion in some punctured disc $\Delta' = \{z \in \mathbb{C} : 0 < |z-\alpha| < r\}$ $(r > 0)$ about α. For $0 < |z-\alpha| < r$ this Laurent expansion is of the form

$$\begin{aligned} f(z) &= \frac{a_{-k}}{(z-\alpha)^k} + \frac{a_{-k+1}}{(z-\alpha)^{k-1}} + \cdots \\ &= \frac{1}{(z-\alpha)^k}\left(a_{-k} + a_{-k+1}(z-\alpha) + \cdots\right) \\ &= \frac{1}{(z-\alpha)^k}\, g(z), \end{aligned}$$

say, where g is analytic in $\Delta = \Delta' \cup \{\alpha\}$ and $g(\alpha) = a_{-k} \neq 0$. Since $g(\alpha) \neq 0$ it follows by continuity that g is analytic and non-zero in some neighbourhood $D \subseteq \Delta$ of α. So if f has a pole of order k at α then, we can express $f(z)$ in the form

$$f(z) = \frac{g(z)}{(z-\alpha)^k}$$

in some punctured disc $D' = D \setminus \{\alpha\}$ about α, where g is analytic and non-zero in D.
(ii) "<u>If</u>" Suppose that

$$f(z) = \frac{g(z)}{(z-\alpha)^k} \tag{1}$$

in D' where, g is analytic and non-zero in the disc $D = D' \cup \{\alpha\}$. Then $g(z)$ has a Taylor series expansion

$$g(z) = c_0 + c_1(z-\alpha) + c_2(z-\alpha)^2 + \cdots + c_k(z-\alpha)^k + c_{k+1}(z-\alpha)^{k+1} + \cdots$$

valid in D where, $c_0 = g(\alpha) \neq 0$. It follows from (1) that

$$f(z) = \frac{g(z)}{(z-\alpha)^k} = \frac{c_0}{(z-\alpha)^k} + \frac{c_1}{(z-\alpha)^{k-1}} + \cdots + c_k + c_{k+1}(z-\alpha) \cdots$$

in D' where, $c_0 \neq 0$. Hence the function f has a pole of order k at α. $\square$

This result suggests that there is a connection between zeros and poles. This is indeed true as the next result illustrates.

Theorem 11.8 *If the function f has a zero of order k at α, then $\frac{1}{f}$ has a pole of order k at α.*

Proof. If f has a zero of order k at α, then we can express $f(z)$ in the form

$$f(z) = (z-\alpha)^k g(z), \tag{1}$$

in some disc $D = \{z \in \mathbb{C} : |z-\alpha| < R\}$, $R > 0$, where g is analytic and non-zero in D. So the function $h = \frac{1}{g}$ is **analytic and non-zero in** D. Hence in the punctured disc $D' = D \setminus \{\alpha\}$,

$$\frac{1}{f(z)} = \frac{1}{(z-\alpha)^k} \frac{1}{g(z)} = \frac{h(z)}{(z-\alpha)^k} .$$

Hence f has a pole of order k at α. $\square$

Corollary 11.9 *If the function f has a zero of order m at α and the function g has a zero of order n at α, then*
(i) $\frac{f}{g}$ has a pole of order $(n-m)$ at α if $n > m$;
(i) $\frac{f}{g}$ has a removable singularity at α if $n \leq m$.

Proof. Since the functions f and g have zeros of order m and n respectively at α, We can write

$$f(z) = (z-\alpha)^m h(z) \qquad \text{and} \qquad g(z) = (z-\alpha)^m k(z) \tag{1}$$

in some disc $D = \{z \in \mathbb{C} : |z-\alpha| < r\}$ $(r > 0)$ about α, where the functions h and k are analytic and non-zero in D.
From (1) we see that, for $0 < |z-\alpha| < r$,

$$\frac{f(z)}{g(z)} = \frac{(z-\alpha)^m\, h(z)}{(z-\alpha)^n\, k(z)} = \frac{(z-\alpha)^m}{(z-\alpha)^n}\, j(z), \tag{2}$$

where $j = \frac{h}{k}$ is analytic and non-zero in D.
Case(i) If $n > m$, we see from (2) that,

$$\frac{f(z)}{g(z)} = \frac{j(z)}{(z-\alpha)^{n-m}}$$

in the punctured disc $D' = D \setminus \{\alpha\}$ and $\frac{f}{g}$ has a pole of order $(n-m)$ at α.
Case(ii) If $n \leq m$, we see from (2) that,

$$\frac{f(z)}{g(z)} = (z-\alpha)^{m-n} j(z)$$

in the punctured disc $D' = D \setminus \{\alpha\}$ and $\frac{f}{g}$ has a removable singularity at α. $\square$

Definition 11.10 *If $k = 1$, we say that α is a* ***simple pole****.*

11.6 Examples

1. Find all the singularities in the complex plane of each of the following functions:
(i) $\dfrac{1}{1+z^2}$, (ii) $\dfrac{\sin z}{z^2}$; (iii) $\dfrac{1}{e^z - 1}$;
and classify them.
2. Determine the singularity of $\dfrac{\cot z}{z}$ at the origin.
3. Explain how Laurent expansions are used to classify isolated singularities.

   Find all the singularities in the complex plane of each of the following functions and classify them, giving reasons for your answers.

   In each case find the residue at each of the singularities:
(i) $\cos\left(\dfrac{1}{z-1}\right)$, (ii) $z \cos\left(\dfrac{1}{z-1}\right)$.

Solutions.

1. (i) We see that $\frac{1}{1+z^2}$ is analytic in $\mathbb{C} \setminus \{\pm i\}$ and so has isolated singularities at $\pm i$.

   Now $1 + z^2 = (z-i)(z+i)$ and so $(1+z^2)$ has zeros of order 1 at the points $\pm i$. Thus $\frac{1}{1+z^2}$ has simple poles at $\pm i$.

   (ii) We see that $\frac{\sin z}{z^2}$ is analytic on $\mathbb{C} \setminus \{0\}$ and so 0 is an isolated singularity. Now $\sin z$ has a zero of order 1 at the origin and z^2 has a zero of order 2 at the origin. Hence by Corollary 11.9 $\frac{\sin z}{z^2}$ has a simple pole at the origin.

   (iii) We first note that $e^z = 1$ if and only if $z = 2n\pi i$, where n is an integer. Let $f(z) = e^z - 1$. Then $f(2n\pi i) = 0$, $f'(2n\pi i) = [e^z]_{z=2n\pi i} = e^{2n\pi i} = 1 \neq 0$. Thus f

has zeros of order 1 at the points $2n\pi i (n \in \mathbb{Z})$.

Hence $\frac{1}{f}$ has isolated singularities at the points $2n\pi i (n \in \mathbb{Z})$ and these are simple poles by Theorem 11.8

2. Let

$$h(z) = \frac{\cot z}{z} = \frac{\cos z}{z \sin z} = \frac{f(z)}{g(z)} \qquad \text{where} \quad f(z) = \cos z\,, \quad g(z) = z \sin z\,.$$

Then the function h is analytic on the punctured disc $\{z \in \mathbb{C} : 0 < |z| < \pi\}$ and so h has a isolated singularity at the origin. Now $\sin z$ has a zero of order 1 at the origin and, therefore, g has a zero of order 2 at the origin. By theorem 11.8, $\frac{1}{g}$ has a pole of order 2 at the origin. Hence $h = \frac{f}{g}$ has a pole of order 2 i.e. a double pole at the origin since f is analytic in $\mathbb{C}$ and $f(0) = \cos 0 \neq 0$.

3. Suppose that f has an isolated singularity at α. Then $f(z)$ has a Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-\alpha)^n = \sum_{n=1}^{\infty} \frac{b_n}{(z-\alpha)^n} + \sum_{n=0}^{\infty} a_n (z-\alpha)^n\,,$$

valid in some punctured disc about α, where $b_n = a_{-n}$ for all positive integers n.

A. REMOVABLE SINGULARITY.

If $a_n = 0$ for all $n < 0$, (so that $b_n = 0$ for all positive integers n, i.e. all the negative powers of $(z - \alpha)$ in the Laurent expansion have coefficient 0) then α is called a *removable singularity*.

B. ISOLATED ESSENTIAL SINGULARITY.

If $a_n \neq 0$ for infinitely many $n < 0$, (so that infinitely many of the coefficients b_n are **non-zero** i.e. there are infinitely many negative powers of $(z - \alpha)$ with non-zero coefficient in the Laurent expansion) then f has an *isolated essential singularity* at α.

C. POLE.

If we can find $k \in \mathbb{N}$ such that $a_{-k} \neq 0$ and $a_n = 0$ for $n < -k$, then f is said to have a *pole of order k at α*.

Let

$$g(z) \;=\; \cos\left(\frac{1}{z-1}\right) \qquad\qquad h(z) \;=\; z\cos\left(\frac{1}{z-1}\right).$$

Then g and h are both analytic in $\mathbb{C}\backslash\{1\}$. For $z \neq 1$,

$$g(z) \;=\; \cos\left(\frac{1}{z-1}\right) \;=\; 1 - \frac{1}{2!}\left(\frac{1}{z-1}\right)^2 \;+\; \frac{1}{4!}\left(\frac{1}{z-1}\right)^4 \;-\; \cdots\,,$$

and

$$\begin{aligned}
h(z) &= [(z-1)+1]\cos\left(\frac{1}{z-1}\right)\\
&= [(z-1)+1]\left[1-\frac{1}{2!}\left(\frac{1}{z-1}\right)^2 + \frac{1}{4!}\left(\frac{1}{z-1}\right)^4 - \cdots\right]\\
&= 1+(z-1)-\frac{1}{2!}\left(\frac{1}{z-1}\right)^2-\frac{1}{2!}\left(\frac{1}{z-1}\right)+\frac{1}{4!}\left(\frac{1}{z-1}\right)^4+\frac{1}{4!}\left(\frac{1}{z-1}\right)^3-\cdots .
\end{aligned}$$

These are the Laurent expansions of $g(z)$ and $h(z)$ about the isolated singularity at 1. They are both valid in $\mathbb{C}\backslash\{1\}$ and

$$\mathrm{Res}\{g;1\}=0\,, \qquad\qquad \mathrm{Res}\{h;1\}=-\tfrac{1}{2!}=-\tfrac{1}{2}.$$

11.7 Quick ways of calculating residues at poles

Theorem 11.11 *1. Suppose that f has a pole of order k at α. Then*

$$\mathrm{Res}\,\{f\,;\alpha\}=\frac{1}{(k-1)!}\lim_{z\to\alpha}\frac{d^{k-1}}{dz^{k-1}}[(z-\alpha)^k f(z)]$$

2. If $f=\frac{g}{h}$ where g and h are analytic at α and $g(\alpha)\neq 0$, $h(\alpha)=0$, $h'(\alpha)\neq 0$ (so h has a simple zero at α), then f has a simple pole at α and

$$\mathrm{Res}\,\{f\,;\alpha\} = \frac{g(\alpha)}{h'(\alpha)}\,.$$

Proof.

1. Since f has a pole of order k at α, it is analytic in some punctured disc $D^*=\{z\in\mathbb{C}:0<|z-\alpha|<r\}$ for some $r>0$. In D^*, $f(z)$ has a Laurent expansion of the form

$$f(z)=\frac{a_{-k}}{(z-\alpha)^k}+\frac{a_{-k+1}}{(z-\alpha)^{(k-1)}}+\cdots+\frac{a_{-1}}{(z-\alpha)}+\sum_{n=0}^{\infty}a_n(z-\alpha)^n\,,$$

where $a_{-k}\neq 0$ since the pole is of order k. Thus in D^*, i.e. for $0<|z-\alpha|<r$, we have

$$(z-\alpha)^k f(z)=a_{-k}+a_{-k+1}(z-\alpha)+\cdots+a_{-1}(z-\alpha)^{k-1}+\cdots .$$

Differentiate $k-1$ times to get

$$\frac{d^{k-1}}{dz^{k-1}}\left[(z-\alpha)^k f(z)\right]=(k-1)!\,a_{-1}+k!\,a_0(z-\alpha)+\cdots$$

and let $z\to\alpha$ to get

$$\mathrm{Res}\,\{f\,;\alpha\}=a_{-1}=\frac{1}{(k-1)!}\lim_{z\to\alpha}\frac{d^{k-1}}{dz^{k-1}}\left[(z-\alpha)^k f(z)\right].$$

<u>Note</u>. This shortcut will work when $f(z)$ is given by a formula involving a fraction with an obvious factor $(z-\alpha)^n$ in the <u>denominator</u>.
In other cases you will need to find a Laurent expansion and pick out the coefficient a_{-1} .

2. As $h(\alpha)=0$, $h'(\alpha)\neq 0$, h has a simple zero at $z=\alpha$. Thus $\frac{1}{h}$ has a simple pole at α. As g is analytic and non-zero at α, it follows that $\frac{g}{h}$ has a simple pole at α. Use (1) to find the residue at α with $k=1$. Then, using the relation $h(\alpha)=0$, and the algebra of limits,

$$\operatorname{Res}\{\tfrac{g}{h}\,;\alpha\}=\lim_{z\to\alpha}\frac{d^0}{dz^0}\left[(z-\alpha)\frac{g(z)}{h(z)}\right]=\lim_{z\to\alpha}\frac{(z-\alpha)g(z)}{(h(z)-h(\alpha))}=\lim_{z\to\alpha}\frac{g(z)}{\frac{h(z)-h(\alpha)}{z-\alpha}}=\frac{g(\alpha)}{h'(\alpha)}\,.$$

□

These are important results, which you will find extremely useful when solving problems. They often provide an easy alternative to finding a residue by using a Laurent expansion and should be remembered if you wish to avoid a lot of unnecessary hard work. You have been warned!!!

11.8 Examples

Find the singularities in the complex plane of the following and calculate the residues at each of them:
(i) $\dfrac{1}{1+z^2}$, (ii) $\dfrac{1}{e^z-1}$, (iii) $\dfrac{e^z}{(z-1)^2}$, (iv) $\dfrac{1}{(1+z^2)^9}$, (v) $\dfrac{\sin z}{z^{10}}$.

Solutions.
(i) Let $f(z)=\frac{1}{1+z^2}$. We know that f is analytic in $\mathbb{C}\setminus\{\pm i\}$ and has simple poles at $\pm i$ (see examples - section 11.8). Take

$$g(z)=1\,,\quad h(z)=1+z^2\,,\qquad \text{so that}\quad h'(z)=2z\,,\quad g(i)=1\,,\quad h(i)=0\,,\quad h'(i)=2i\,.$$

Thus

$$\operatorname{Res}\{f;i\}=\frac{g(i)}{h'(i)}=\frac{1}{2i}$$

by Theorem 11.11 part (2). Similarly,

$$\operatorname{Res}\{f;-i\}=\frac{g(-i)}{h'(-i)}=-\frac{1}{2i}.$$

(ii) Let $k(z)=\frac{1}{e^z-1}$. We know that k is analytic in $\mathbb{C}$ except for simple poles at the points $2n\pi i \quad (n\in\mathbb{Z})$ (see examples - section 11.8) . Thus by Theorem 11.11 part (2),

$$\operatorname{Res}\{k;2n\pi i\}=\left[\frac{1}{\frac{d}{dz}\,(e^z-1)}\right]_{z=2n\pi i}=\left[\frac{1}{e^z}\right]_{z=2n\pi i}=1.$$

(iii) Let $m(z) = \frac{e^z}{(z-1)^2}$. Then m is analytic on $\mathbb{C}$ except for a singularity at the point 1. Now $(z-1)^2 e^{-z}$ has a zero of order 2 at $z = 1$ and so its reciprocal m has a double pole at 1. Using Theorem 11.11 part (1) with $k = 2$ and $\alpha = 1$ gives,

$$\operatorname{Res}\{m : 1\} = \frac{1}{(2-1)!} \lim_{z \to 1} \frac{d}{dz}\left[(z-1)^2 \frac{e^z}{(z-1)^2}\right] = \lim_{z \to 1} \frac{d(e^z)}{dz} = e.$$

(iv) Let $p(z) = \frac{1}{(1+z^2)^9}$. Then the function p is analytic on $\mathbb{C}$ except for isolated singularities at $\pm i$.
Since $(1+z^2)^9 = (z+i)^9(z-i)^9$ has zeros of order 9 at $\pm i$, we see that $\frac{1}{(1+z^2)^9}$ has poles of order 9 at $\pm i$, i.e. m has poles of order 9 at $\pm i$. Using Theorem 11.11 part (1) with $k = 9$ gives,

$$\begin{aligned}
\operatorname{Res}\{p\,;i\} &= \frac{1}{8!} \lim_{z \to i} \frac{d^8}{dz^8}\left[(z-i)^9 \frac{1}{(1+z^2)^9}\right] = \frac{1}{8!} \lim_{z \to i} \frac{d^8}{dz^8} \frac{1}{(z+i)^9} \\
&= \frac{1}{8!} \frac{(-9)(-10)(-11)(-12)(-13)(-14)(-15)(-16)}{(i+i)^{17}} \\
&= \frac{16!}{(8!)^2 2^{17} i} .
\end{aligned}$$

Similarly,

$$\begin{aligned}
\operatorname{Res}\{p\,;-i\} &= \frac{1}{8!} \lim_{z \to -i} \frac{d^8}{dz^8}\left[(z+i)^9 \frac{1}{(1+z^2)^9}\right] = \frac{1}{8!} \lim_{z \to -i} \frac{d^8}{dz^8} \frac{1}{(z-i)^9} \\
&= \frac{1}{8!} \frac{(-9)(-10)(-11)(-12)(-13)(-14)(-15)(-16)}{(-i-i)^{17}} \\
&= -\frac{16!}{(8!)^2 2^{17} i} .
\end{aligned}$$

(v) Let $q(z) = \frac{\sin z}{z^{10}}$. Then q is analytic in $\mathbb{C}$ except for an isolated singularity. Now $\sin z$ has a zero of order 1 at the origin and z^{10} has a zero of order 10 at the origin. From Corollary 11.9, we see that q has a pole of order 9 at the origin and the shortcut won't be easy to use. In this case it will be easier to use a Laurent expansion. For $z \neq 0$,

$$\frac{\sin z}{z^{10}} = \frac{1}{z^9} - \frac{1}{3!z^7} + \frac{1}{5!z^5} - \frac{1}{7!z^3} + \frac{1}{9!z} - \frac{z}{11!} + \cdots$$

and $\operatorname{Res}\{q; 0\}$ is clearly $\frac{1}{9!}$.

Note that Theorem 11.11 part(1) is very messy to apply on this problem.

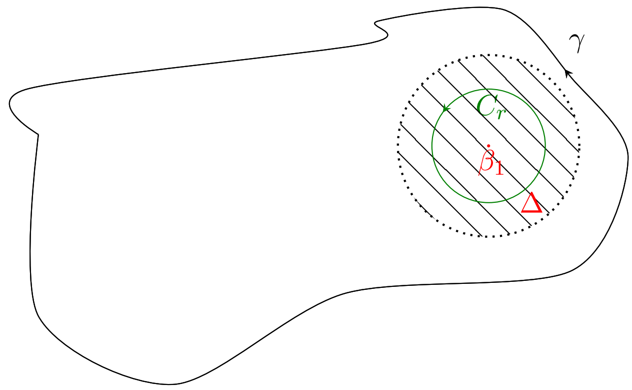
12 The Residue Theorem

Let D be a simply connected region containing a simple positively oriented contour γ. Suppose f is analytic on D except for finitely many singularities $\beta_1, \ldots, \beta_n$, none of which lie on γ. Then

$$\int_\gamma f(z)dz = 2\pi i \times (\text{sum of residues of } f \text{ at the singularities of } f \text{ inside } \gamma).$$

Proof. (i) Suppose γ contains exactly one singularity (at β_1). Then there is some disc $\Delta = \{z \in \mathbb{C} : |z - \beta_1| < R\}$ inside γ. Thus f is analytic in the punctured disc

$$\Delta' = \Delta \setminus \{\beta_1\} = \{z \in \mathbb{C} : 0 < |z - \beta_1| < R\}.$$



Hence $f(z)$ has a Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - \beta_1)^n$$

valid in Δ', where

$$a_n = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{(z - \beta_1)^{n+1}}\, dz \quad (n = 0, \pm 1, \pm 2, \cdots)$$

and C_r is any circular contour $z = \beta_1 + re^{it}$ $(0 \le t \le 2\pi)$ with $0 < r < R$. In particular $a_{-1} = \frac{1}{2\pi i} \int_{C_r} f(z)\, dz$

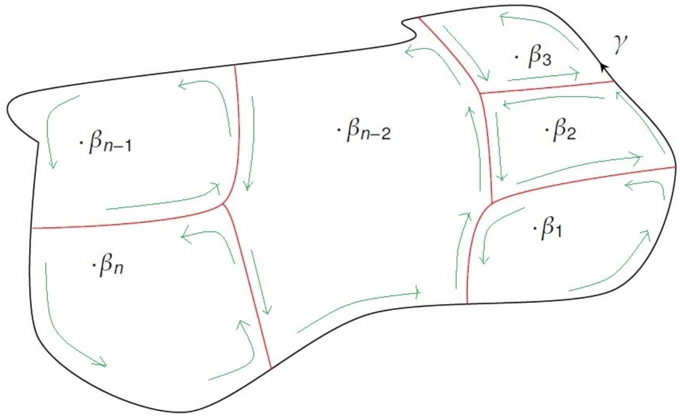
$$\text{i.e.} \quad \text{Res}\{f; \beta_1\} = \frac{1}{2\pi i} \int_{C_r} f(z)\, dz.$$

Now the function f is analytic on a region containing C_r, γ and the region between them. By Theorem 9.1

$$\int_\gamma f(z)\, dz = \int_{C_r} f(z)\, dz = 2\pi i\, \text{Res}\{f; \beta_1\}\,. \qquad (1)$$

(ii) Now suppose γ contains a finite number of singularities at $\beta_1, \beta_2, \cdots, \beta_n$. Draw extra paths as in the diagram below, to produce n simple contours $\gamma_1, \gamma_2, \cdots \gamma_n$, such that the contour γ_r contains exactly one singularity (viz. the one at β_r) for $r = 1, 2, \cdots, n$ and

$$\int_\gamma f(z)\, dz = \sum_{r=1}^{n} \int_{\gamma_r} f(z)\, dz.$$



Using the previous part, we see from (1), that

$$\int_{\gamma_r} f(z)\,dz = 2\pi i\,\mathrm{Res}\{f;\beta_r\}\,.$$

Hence

$$\int_{\gamma} f(z)\,dz = 2\pi i\sum_{r=1}^{n}\mathrm{Res}\{f;\beta_r\}\,.$$

$\square$

12.1 Examples

1. Evaluate $\displaystyle\int_c \frac{dz}{z^2(z-3)}$ where $c: z = 71e^{it}$ $(0 \le t \le 2\pi)$.

2. Evaluate $\displaystyle\int_c \frac{dz}{z^4+1}$, where c denotes the semi-circular contour consisting of the straight line from -2 to 2 along the real axis, followed by the semicircle $z = 2e^{it}$ $(0 \le t \le \pi)$ of radius 2 in the upper half plane from 2 back to -2.

3. Let γ be the square contour with vertices -3, $-3i$, 3, $3i$ described in the anti-clockwise direction. Evaluate

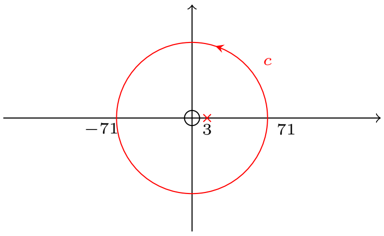
$$\text{(i)}\ \int_\gamma z^3\cos(1/z)\,dz\,,\qquad \text{(ii)}\ \int_\gamma \cos(1/z)\,dz\,.$$

Solutions.

1. Let

$$f(z) = \frac{1}{z^2(z-3)}\,.$$

Then f is analytic in $\mathbb{C}\setminus\{0,3\}$. Now $z^2(z-3)$ has a zero of order 2 at the origin and a simple zero at the point $z=3$.



Hence f has a double pole at the origin and a simple pole at $z = 3$. Both these singularities lie inside the contour c, which is a simple contour described in the positive direction. Now

$$\begin{aligned}
\operatorname{Res}\{f;0\} &= \frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}\left(z^2 \times \frac{1}{z^2(z-3)}\right) = \left(\frac{-1}{(z-3)^2}\right)_{z=0} = -\frac{1}{9},\\
\operatorname{Res}\{f;3\} &= \left(\frac{\frac{1}{z^2}}{\frac{d}{dz}(z-3)}\right)_{z=3} = \frac{1}{9}
\end{aligned}$$

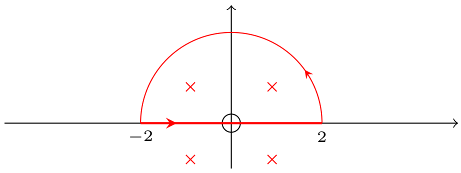
By Cauchy's Residue Theorem

$$\int_c \frac{dz}{z^2(z-3)} = 2\pi i\left(\operatorname{Res}\{f;0\} + \operatorname{Res}\{f;3\}\right) = 0.$$

2. Let

$$g(z) = \frac{1}{z^4+1}\,.$$

Then g is analytic in $\mathbb{C}$ except for the four points at which $z^4+1=0$. Now z^4+1 has simple zeros at the points $\exp\left(\pm\frac{\pi i}{4}\right), \exp\left(\pm\frac{3\pi i}{4}\right)$ and, hence, g has simple poles, at $\exp\left(\pm\frac{\pi i}{4}\right), \exp\left(\pm\frac{3\pi i}{4}\right)$.



The simple poles at $\exp\left(\frac{\pi i}{4}\right), \exp\left(\frac{3\pi i}{4}\right)$ lie inside the contour whereas the other two lie in the lower half plane, so are outside the contour. Moreover c is a simple contour described in the positive direction. Now

$$\operatorname{Res}\{g\,;\exp\left(\tfrac{\pi i}{4}\right)\} = \left(\frac{1}{\frac{d}{dz}\left(z^4+1\right)}\right)_{z=\exp\left(\frac{\pi i}{4}\right)} = \left(\frac{1}{4z^3}\right)_{z=\exp\left(\frac{\pi i}{4}\right)} = \frac{1}{4e^{\frac{3\pi i}{4}}} = \frac{-e^{\frac{\pi i}{4}}}{4}\;;$$

$$\operatorname{Res}\{g\,;\exp\left(\tfrac{3\pi i}{4}\right)\} = \left(\frac{1}{\frac{d}{dz}\left(z^4+1\right)}\right)_{z=\exp\left(\frac{3\pi i}{4}\right)} = \left(\frac{1}{4z^3}\right)_{z=\exp\left(\frac{3\pi i}{4}\right)} = \frac{1}{4e^{\frac{9\pi i}{4}}} = \frac{e^{\frac{-\pi i}{4}}}{4}\,.$$

Thus the sum of the residues is

$$\mathrm{Res}\{g\,;\exp\left(\tfrac{\pi i}{4}\right)\} + \mathrm{Res}\{g\,;\exp\left(\tfrac{3\pi i}{4}\right)\} = \tfrac{1}{4}(e^{-\frac{\pi i}{4}} - e^{\frac{\pi i}{4}}) = -\tfrac{1}{4}.2i\sin\tfrac{\pi}{4} = \tfrac{-i}{2\sqrt{2}}.$$

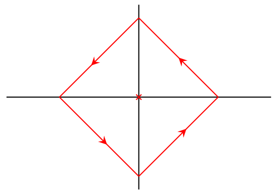
By Cauchy's Residue Theorem

$$\int_c \frac{dz}{z^4+1} = 2\pi i \times \frac{-i}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}}\ .$$

3. Let

$$h(z) = z^3 \cos(1/z), \qquad k(z) = \cos(1/z).$$

Then h and k are both analytic in $\mathbb{C} \setminus \{0\}$ and have isolated essential singularities at the origin.



Now the origin lies inside the contour γ, which is a simple contour described in the positive direction. Now. for $z \neq 0$,

$$\begin{aligned} z^3 \cos(1/z) &= z^3 \left(1 - \frac{1}{z^2\, 2!} + \frac{1}{z^4\, 4!} - \cdots\right) = z^3 - \frac{z}{2!} + \frac{1}{z\, 4!} - \cdots, \\ \cos(1/z) &= \left(1 - \frac{1}{z^2\, 2!} + \frac{1}{z^4\, 4!} - \cdots\right). \end{aligned}$$

Thus $\operatorname{Res}\{h; 0\} = \frac{1}{4!} = \frac{1}{24}$ and $\operatorname{Res}\{k; 0\} = 0$.

(i) By Cauchy's Residue Theorem $\int_\gamma h(z)\, dz = \int_\gamma z^3 \cos(1/z)\, dz = \frac{2\pi i}{24} = \frac{\pi i}{12}$.

(ii) By Cauchy's Residue Theorem $\int_\gamma k(z)\, dz = \int_\gamma \cos(1/z)\, dz = 0$.

12.2 Application to the evaluation of certain real integrals

Now we explain how complex integration helps us evaluate certain real integrals.

Integrals of the form $\int_{-\infty}^{\infty} \phi(x) \cos \lambda x\, dx$, $\int_{-\infty}^{\infty} \phi(x) \sin \lambda x\, dx$, $\int_0^{\infty} \phi(x) \cos \lambda x\, dx$, $\int_0^{\infty} \phi(x) \sin \lambda x\, dx$, where ϕ is a rational function and λ is a positive real number

Integrals of this form are evaluated by integrating

$$f(z) = \phi(z)\, e^{i\lambda z}$$

around a suitable contour. The method is illustrated in the following example.

Example. Show that

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\, dx = \frac{\pi}{e}.$$

Deduce that

$$\int_0^{\infty} \frac{\cos x}{x^2+1}\, dx = \frac{\pi}{2e}.$$

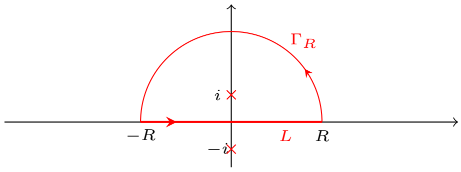
Solution.

Let

$$p(z) = 1, \quad q(z) = z^2 + 1, \quad \phi(z) = \frac{p(z)}{q(z)}, \quad f(z) = \phi(z) e^{iz} = \frac{1}{z^2+1}\, e^{iz}.$$

Then the function f is analytic in $\mathbb{C}$ except for simple poles at $\pm i$.

We use the contour γ shown below



consisting of the straight line segment L from $-R$ to R followed by the semi-circle Γ_R given by $z = Re^{it}(0 \leq t \leq \pi)$, where $R > 1$. Now f is analytic in $\mathbb{C}$ except for simple poles at $\pm i$, neither of which lie on γ. By the Cauchy's Residue Theorem

$$\int_\gamma f(z)\,dz = 2\pi i \text{Res}\{f; i\}\,, \tag{1}$$

since the pole at i is inside γ and the pole at $-i$ is outside. Now

$$\text{Res}\{f; i\} = \left[\frac{e^{iz}}{\frac{d}{dz}(z^2+1)}\right]_{z=i} = \frac{e^{-1}}{2i} = \frac{1}{2ie}.$$

By (1)

$$\begin{aligned} \int_L f(z)\,dz + \int_{\Gamma_R} f(z)\,dz &= \int_\gamma f(z)\,dz = \frac{2\pi i}{2ie} = \frac{\pi}{e} \\ \text{i.e. } \int_{-R}^{R} f(x)\,dx + \int_{\Gamma_R} f(z)\,dz &= \frac{\pi}{e}. \end{aligned} \tag{2}$$

We now show that $\int_{\Gamma_R} f(z)\,dz \to 0$ as $R \to \infty$.
By definition

$$\begin{aligned} \left|\int_{\Gamma_R} f(z)\,dz\right| &= \left|\int_0^\pi \frac{1}{R^2e^{2it}+1}\, e^{iR(\cos t + i\sin t)}\, iRe^{it}\,dt\right| \\ &\leq \int_0^\pi \left|\frac{iRe^{it}}{R^2e^{2it}+1}\, e^{iR(\cos t+i\sin t)}\right| dt \\ &\leq \int_0^\pi \frac{R}{R^2-1}\, e^{-R\sin t}\,dt \\ &\leq \int_0^\pi \frac{R}{R^2-1}\,dt = \frac{\pi R}{R^2-1}\,. \end{aligned}$$

Thus $\left|\int_{\Gamma_R} f(z)\,dz\right| \to 0$ as $R \to \infty$.
Letting $R \to \infty$ in (2) gives

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x^2+1}\, e^{ix}\,dx = \frac{\pi}{e}$$

and equating real parts gives

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{\cos x}{1+x^2}\,dx=\frac{\pi}{e}.$$

Since we know that $\int_{-\infty}^{\infty}\frac{\cos x}{x^2+1}\,dx$ exists the limit above gives its value

$$\text{i.e.}\qquad \int_{-\infty}^{\infty}\frac{\cos x}{x^2+1}\,dx=\frac{\pi}{e}.$$

Using the substitution $t=-x$ we see that

$$\int_{-\infty}^{0}\frac{\cos x}{1+x^2}\,dx=\int_{0}^{\infty}\frac{\cos(-t)}{1+t^2}\,dt=\int_{0}^{\infty}\frac{\cos t}{1+t^2}\,dt.$$

Thus

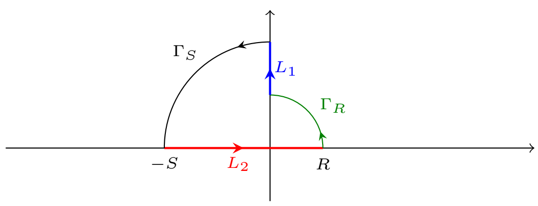
$$\int_{0}^{\infty}\frac{\cos x}{1+x^2}\,dx=\frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos x}{1+x^2}\,dx=\frac{\pi}{2e}.$$

Notes.

1. This method gives $\lim\limits_{R\to\infty}\int_{-R}^{R}f(x)\,dx$. However $\int_{-\infty}^{\infty}f(x)\,dx$ is defined to be

   $$\lim_{R,\ S\to\infty}\int_{-S}^{R}f(x)\,dx$$

   provided the limit exists and is finite, where $R,S\to\infty$ **independently**. Thus the existence of $\lim_{R\to\infty}\int_{-R}^{R}f(x)\,dx$ does not guarantee the existence of $\int_{-\infty}^{\infty}f(x)\,dx$. For example, $\int_{-R}^{R}x\,dx=0$ and so $\lim_{R\to\infty}\int_{-R}^{R}f(x)\,dx=0$, but $\int_{-\infty}^{\infty}x\,dx$ does not exist.
   We can get round this problem by modifying the contour and using the contour shown below.

   We would, however have to show that $\int_{\Gamma_R}f(z)\,dz\to0$ as $R\to\infty$, $\int_{\Gamma_S}f(z)\,dz\to0$ as $S\to\infty$ and $\int_{L_1}f(z)\,dz\to0$ as $R,S\to\infty$.

2. If we do several examples, we will have to use the same method again and again to deal with the integrals along the circular arcs and the part of the imaginary axis. What is called for here is a shortcut. The next theorem will provide this. This will save you a lot of work in practice , **but you must remember that the underlying idea is to evaluate the real integral by integrating a suitable complex function round a suitable contour using the Residue Theorem.**

Theorem 12.1 *Suppose that*

$$\phi(z) = \frac{p(z)}{q(z)} \quad \textit{and} \quad f(z) = \phi(z)\, e^{i\lambda z} ,$$

where $\lambda \in \mathbb{R}$, and p and q are polynomials with no common factor other than 1. Suppose also that the polynomial q is ***non-zero on the real axis*** *(i.e. none of the singularities of the rational function ϕ lie on the real axis.)*

If the degree of the polynomial q is greater that the degree of the polynomial p and $\lambda > 0$, then

$$\int_{-\infty}^{\infty} \phi(x) e^{i\lambda x}\, dx = 2\pi i \sum_{r=1}^{k} \operatorname{Res}\{f\,; z_r\}\,, \tag{1}$$

where z_1, z_2, $\cdots$ z_k are the zeros of the polynomial q in the ***upper half-plane*** *$H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ (i.e. z_1, z_2, $\cdots$ z_k are the singularities of f in the upper half-plane H.)*

From (1)

$$\int_{-\infty}^{\infty} \phi(x) \cos \lambda x\, dx = \operatorname{Re}\left(2\pi i \sum_{r=1}^{k} \operatorname{Res}\{f\,; z_r\}\right) ,$$

and

$$\int_{-\infty}^{\infty} \phi(x) \sin \lambda x\, dx = \operatorname{Im}\left(2\pi i \sum_{r=1}^{k} \operatorname{Res}\{f\,; z_r\}\right) .$$

Before proving this result we need two Lemmas, which are given below.

Lemma 12.2 *For $0 < \theta \leq \frac{\pi}{2}$,*

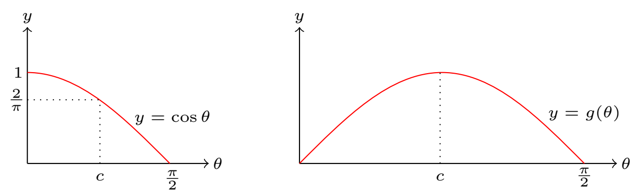
$$\frac{2}{\pi} \leq \frac{\sin\theta}{\theta}.$$

Proof. Let

$$g(\theta) = \sin\theta - \tfrac{2}{\pi}\,\theta.$$

Then

$$g'(\theta) = \cos\theta - \tfrac{2}{\pi} \tag{2}$$



Now let $c = \cos^{-1}(\frac{2}{\pi})$. Then $0 < c < \frac{\pi}{2}$ and $g'(\theta) \geq 0$ when $0 \leq \theta \leq c$. Thus g is increasing on $[0, c]$ and so $g(\theta) \geq g(0) = 0$ on $[0, c]$ i.e. $\frac{\sin\theta}{\theta} \geq \frac{2}{\pi}$ on $(0, c]$.

Similarly g is decreasing on $[c, \frac{\pi}{2}]$ because $g'(\theta) \leq 0$ on $[c, \frac{\pi}{2}]$ and so $g(\theta) \geq g(\frac{\pi}{2}) = 0$ on $[c, \frac{\pi}{2}]$ giving $\frac{\sin\theta}{\theta} \geq \frac{2}{\pi}$ on $(c, \frac{\pi}{2}]$. $\square$

Corollary 12.3 *For any positive real number R,*

$$-R\sin\theta \le -\frac{2R\theta}{\pi} \qquad (0 < \theta \le \frac{\pi}{2}).$$

Lemma 12.4 *Suppose that the function ϕ is analytic on the region $D = \{z \in \mathbb{C} : |z| > r\}$ for some $r > 0$. Let Γ_R be given by*

$$z = Re^{it}\,(\theta_1 \le t \le \theta_2)\,,$$

where $0 \le \theta_1 < \theta_2 \le \pi$ and $R > r$. Let $M(R)$ be the maximum value of $|\phi(z)|$ on Γ_R and let λ be a positive real number. If $M(R) \to 0$ as $R \to \infty$, then

$$I_R = \int_{\Gamma_R} \phi(z)\, e^{i\lambda z}\, dz \to 0 \text{ as } R \to \infty\,.$$

Proof. Using corollary 12.3, we see that

$$\begin{array}{rclrl}
|I_R| & = & \left|\int_{\Gamma_R} \phi(z)\, e^{i\lambda z}\, dz\right| & = & \left|\int_{\theta_1}^{\theta_2} \phi(Re^{it})\, e^{i\lambda R(\cos t + i\sin t)} iRe^{it}\, dt\right| \\
& \le & \int_{\theta_1}^{\theta_2} \left|\phi(Re^{it})\, e^{i\lambda R(\cos t + i\sin t)} iRe^{it}\right|\, dt & \le & M(R) \int_{\theta_1}^{\theta_2} Re^{-\lambda R\sin t}\, dt \\
& \le & R\, M(R) \int_0^{\pi} e^{-\lambda R\sin t}\, dt & = & 2R\, M(R) \int_0^{\frac{\pi}{2}} e^{-\lambda R\sin t}\, dt \\
& \le & 2R\, M(R) \int_0^{\frac{\pi}{2}} e^{-\frac{2\lambda Rt}{\pi}}\, dt & = & M(R) \left[-\frac{\pi}{\lambda} e^{-\frac{2\lambda Rt}{\pi}}\right]_0^{\frac{\pi}{2}} \\
& = & \frac{\pi}{\lambda}\, M(R)[1 - e^{-\lambda R}] \to 0 \text{ as } R \to \infty\,. & &
\end{array}$$

□

Lemma 12.5 *Suppose that the function ϕ is analytic on the region $D = \{z \in \mathbb{C} : |z| > r\}$ for some $r > 0$. Let L_1 be the straight line segment from iR to iS on the imaginary axis, where $R, S > r$. Let λ be a positive real number and let $t \in \mathbb{R}$. If $|\phi(it)| \to 0$ as $t \to \infty$, then*

$$I_R = \int_{L_1} \phi(z)\, e^{i\lambda z}\, dz \to 0 \text{ as } R, S \to \infty\,.$$

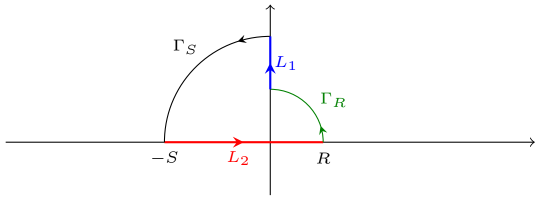
Proof. Since $|\phi(it)| \to 0$ as $t \to \infty$, there exists a real number $r^* \ge r$ such that $|\phi(it)| < 1$ for all $t > r^*$. Thus for $R, S > r^*$,

$$\left|\int_{L_1} \phi(z)e^{i\lambda z}\, dz\right| \le \left|\int_R^S \left|\phi(it)\, e^{-\lambda t}\, i\right|\, dt\right| \le \left|\int_R^S e^{-\lambda t}\, dt\right| = \left|-\frac{1}{n}\left[e^{-\lambda R} - e^{-\lambda S}\right]\right| \to 0$$

as $R, S \to \infty$. □

We now return to the proof of Theorem 12.1.

Proof. (Theorem 12.1) Suppose that all the zeros of the polynomial q lie in the closed disc $D^* = \{z \in \mathbb{C} : |z| \le R^*\}$. Let $R, S > R^*$ and let C be the contour shown below.



Thus

$$C = \Gamma_R + L_1 + \Gamma_S + L_2 \,,$$

where Γ_R is given by $z = Re^{it}\,(0 \le t \le \frac{\pi}{2})$, Γ_S is given by $z = Se^{it}\,(\frac{\pi}{2} \le t \le \pi)$, L_1 is the imaginary axis from iR to iS and L_2 is the real axis from $-S$ to R. Hence

$$\int_C f(z)\,dz = \int_{\Gamma_R} f(z)\,dz + \int_{L_1} f(z)\,dz + \int_{\Gamma_S} f(z)\,dz + \int_{L_2} f(z)\,dz \tag{1}$$

By Cauchy's Residue Theorem

$$\int_C f(z)\,dz = 2\pi i \sum_{r=1}^{k} \mathrm{Res}\{f; z_r\}\,. \tag{2}$$

By Lemma 12.4

$$\int_{\Gamma_R} f(z)\,dz \to 0 \text{ as } R \to \infty\,, \quad \int_{\Gamma_S} f(z)\,dz \to 0 \text{ as } S \to \infty\,. \tag{3}$$

By Lemma 12.5

$$\int_{L_1} f(z)\,dz \to 0 \text{ as } R, S \to \infty\,. \tag{4}$$

From (1), (2), (3), (4) we see that

$$\int_{L_2} f(z)\,dz = \int_{-S}^{R} \phi(t)e^{i\lambda t}\,dt \to 2\pi i \sum_{r=1}^{k} \mathrm{Res}\{f; z_r\} \text{ as } R, S \to \infty\,.$$

Thus

$$\int_{-\infty}^{\infty} \phi(t)e^{i\lambda t}\,dt = 2\pi i \sum_{r=1}^{k} \mathrm{Res}\{f; z_r\}\,.$$

Equating real and imaginary parts gives the final results in the theorem. □

12.3 Examples

1. Evaluate $\int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2+2x+5}\,dx$.

2. Evaluate $\int_0^{\infty} \frac{\cos \pi x}{(1+x^2)^2}\,dx$.

3. Let $\alpha > 0$. Evaluate $\int_0^{\infty} \frac{\cos \alpha x}{1+x^2}\,dx$. By using a suitable value for α and a standard result, deduce that

$$\int_0^{\infty} \frac{\cos^2 x}{1+x^2}\,dx = \frac{\pi}{4e^2}\,(1+e^2).$$

Solutions.

1. Let

$$p(z) = z, \quad q(z) = z^2 + 2z + 5, \quad \phi(z) = \frac{p(z)}{q(z)}\,, \quad f(z) = \phi(z)e^{i\pi z} = \frac{ze^{i\pi z}}{z^2+1}\,.$$

Then
(i) $q(z) = z^2 + 2z + 5 = (z+1)^2 + 4$ and so the polynomial q is non-zero on the real axis,
(ii) the degree of the polynomial $q >$ the degree of the polynomial p,
(iii) the function f is analytic in $\mathbb{C}$ except for simple poles at $-1 \pm 2i$. The only singularity of f in the upper half -plane $H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ is the simple pole at $-1 + 2i$.
Now

$$\begin{aligned}\operatorname{Res}\{f; -1+2i\} &= \left(\frac{ze^{i\pi z}}{\frac{d}{dz}\,(z^2+2z+5)}\right)_{z=-1+2i} = \left(\frac{ze^{i\pi z}}{(2z+2)}\right)_{z=-1+2i} \\ &= \frac{(-1+2i)e^{i\pi(-1+2i)}}{4i} = \frac{(-1+2i)e^{-i\pi}\, e^{-2\pi}}{4i} = -\frac{(-1+2i)e^{-2\pi}}{4i}.\end{aligned}$$

By theorem 12.1

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{x}{x^2+2x+5}\, e^{i\pi x}\, dx &= \int_{-\infty}^{\infty} \phi(x)e^{i\pi x}\, dx = 2\pi i \operatorname{Res}\{f; -1+2i\} \\ &= 2\pi i \left[-\frac{(-1+2i)e^{-2\pi}}{4i}\right] = \frac{\pi}{2}\, e^{-2\pi}\,(1-2i).\end{aligned}$$

Equating imaginary parts gives

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+2x+5}\, dx = -\pi\, e^{-2\pi}\,.$$

2. Let

$$p(z) = 1, \quad q(z) = (z^2+1)^2, \quad \phi(z) = \frac{p(z)}{q(z)}\,, \quad f(z) = \phi(z)e^{i\pi z} = \frac{e^{i\pi z}}{(z^2+1)^2}\,.$$

Then
(i) the polynomial q is non-zero on the real axis,
(ii) the degree of the polynomial $q >$ the degree of the polynomial p,
(iii) the function f is analytic in $\mathbb{C}$ except for double poles at $\pm i$. The only singularity of f in the upper half -plane $H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ is the double pole at i.

Now

$$\begin{aligned}
\operatorname{Res}\{f;i\} &= \frac{1}{1!}\lim_{z\to i}\frac{d}{dz}\left((z-i)^2 f(z)\right) &&= \lim_{z\to i}\frac{d}{dz}\left((z-i)^2\frac{e^{i\pi z}}{(z-i)^2(z+i)^2}\right)\\
&= \lim_{z\to i}\frac{d}{dz}\left(\frac{e^{i\pi z}}{(z+i)^2}\right) &&= \lim_{z\to i}\left(\frac{i\pi e^{i\pi z}}{(z+i)^2} - \frac{2\,e^{i\pi z}}{(z+i)^3}\right)\\
&= \frac{i\pi e^{-\pi}}{4i^2} - \frac{2e^{-\pi}}{8i^3} &&= -\frac{i\,e^{-\pi}\,(\pi+1)}{4}\,.
\end{aligned}$$

By Theorem 12.1,

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{1}{(1+x^2)^2}\,e^{i\pi x}\,dx &= \int_{-\infty}^{\infty}\phi(x)e^{i\pi x}\,dx &&= 2\pi i\operatorname{Res}\{f;i\}\\
&= 2\pi i\times\left(-\frac{i\,e^{-\pi}\,(\pi+1)}{4}\right) &&= \frac{\pi}{2}\,e^{-\pi}\,(\pi+1)
\end{aligned}$$

and equating real parts gives

$$\int_{-\infty}^{\infty}\frac{\cos\pi x}{(1+x^2)^2}\,dx = \frac{\pi}{2}\,e^{-\pi}\,(\pi+1).$$

Using the substitution $t = -x$ we see that

$$\int_{-\infty}^{0}\frac{\cos\pi x}{(1+x^2)^2}\,dx = \int_{0}^{\infty}\frac{\cos(-\pi t)}{(1+t^2)^2}\,dt = \int_{0}^{\infty}\frac{\cos\pi t}{(1+t^2)^2}\,dt\,.$$

Thus

$$\int_{0}^{\infty}\frac{\cos\pi x}{(1+x^2)^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos\pi x}{(1+x^2)^2}\,dx = \frac{\pi}{4}\,e^{-\pi}\,(\pi+1).$$

3. Let

$$p(z) = 1,\quad q(z) = z^2+1,\quad \phi(z) = \frac{p(z)}{q(z)}\,,\quad f(z) = \phi(z)e^{i\alpha z} = \frac{e^{i\alpha z}}{z^2+1}\,.$$

Then
(i) the polynomial q is non-zero on the real axis,
(ii) the degree of the polynomial $q >$ the degree of the polynomial p,
(iii) the function f is analytic in $\mathbb{C}$ except for simple poles at $\pm i$. The only singularity of f in the upper half -plane $H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ is the simple pole at i.
(iv) $\alpha > 0$.

Now

$$\operatorname{Res}\{f;i\} = \left(\frac{e^{i\alpha z}}{\frac{d}{dz}\,(z^2+1)}\right)_{z=i} = \frac{e^{-\alpha}}{2i}.$$

By Theorem 12.1

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,e^{i\alpha x}\,dx = \int_{-\infty}^{\infty}\phi(x)e^{i\alpha x}\,dx = 2\pi i\operatorname{Res}\{f;i\}\,) = 2\pi i\times\frac{e^{-\alpha}}{2i} = \pi e^{-\alpha}$$

and equating real parts gives

$$\int_{-\infty}^{\infty} \frac{\cos \alpha x}{1+x^2}\, dx = \pi e^{-\alpha}.$$

Using the substitution $t = -x$ we see that

$$\int_{-\infty}^{0} \frac{\cos \alpha x}{1+x^2}\, dx = \int_{0}^{\infty} \frac{\cos(-\alpha t)}{1+t^2}\, dt = \int_{0}^{\infty} \frac{\cos \alpha t}{1+t^2}\, dt.$$

Thus

$$\int_{0}^{\infty} \frac{\cos \alpha x}{1+x^2}\, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos \alpha x}{1+x^2}\, dx = \frac{\pi e^{-\alpha}}{2}. \tag{1}$$

We notice that

$$\int_{0}^{\infty} \frac{\cos^2 x}{1+x^2}\, dx = \frac{1}{2} \int_{0}^{\infty} \frac{1+\cos 2x}{1+x^2}\, dx. \tag{2}$$

Putting $\alpha = 2$ in equation (1) gives,

$$\int_{0}^{\infty} \frac{\cos 2x}{1+x^2}\, dx = \frac{\pi e^{-2}}{2} = \frac{\pi}{2e^2} \tag{3}$$

and

$$\int_{0}^{\infty} \frac{1}{1+x^2}\, dx = \left[\tan^{-1} x\right]_0^{\infty} = \frac{\pi}{2}. \tag{4}$$

From equations (1), (2), (3), (4) it follows that

$$\int_{0}^{\infty} \frac{\cos^2 x}{1+x^2}\, dx = \frac{\pi}{4} \left(\frac{1}{e^2} + 1\right) = \frac{\pi(1+e^2)}{4e^2}\ .$$