# Previously: Cauchy's Integral Formula and Consequences

## Theorem (Cauchy's integral formula)

*Let $\gamma$ be a simple contour described in the positive direction. Let $w$ lie inside $\gamma$. Suppose that $f$ is analytic on a simply connected region $D$ containing $\gamma$ and its interior. Then:*

$$f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - w} dz.$$

## Uses of Cauchy's Integral Formula

- Calculate contour integrals
- Values of $f$ inside a contour determined by values on boundary
- Proving $f$ has convergent Taylor series!

## Today: Other applications

# Analytic functions have all derivatives!

## Theorem (Cauchy's Integral formula for the derivatives)

*Let $\gamma$ be a simple contour described in the positive direction. let $w$ be any point inside $\gamma$. Suppose $f$ analytic on a simply-connected region $D$ containing $\gamma$. Then*

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-w)^{n+1}} dz$$

## Proof.

Take $\frac{d}{dw}$ of both sides of CIF. Differentiate inside the integral. $\square$

## Example

Let $\gamma$ be the square with vertices $1, i, -1, -i$. Evaluate

$$\int_\gamma \frac{e^z}{z^n} dz$$

# Another application of CIF: Liouville's Theorem

### Theorem (Liouville's Theorem)

*A function which is analytic and bounded in the complex plane is a constant.*

### Proof.

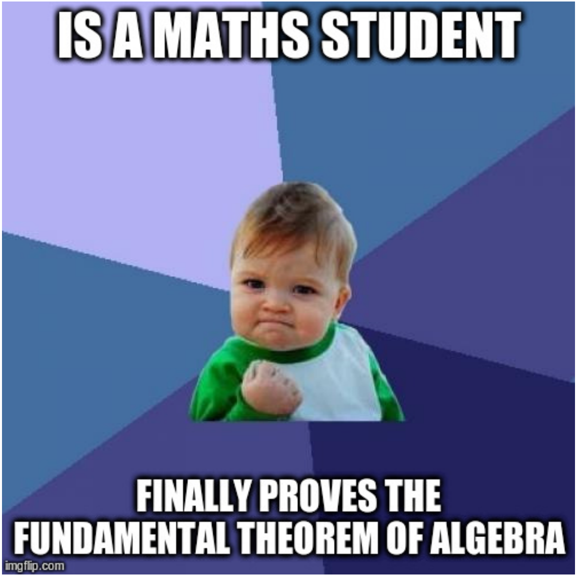
Let $a, b \in \mathbb{C}$.

1. Rewrite $f(a) - f(b)$ as an integral around $|z| = R$ using CIF.
2. Use ML to bound $|f(a) - f(b)|$
3. As $R \to \infty$, the bound goes to 0.

$\square$

Toy applications of Liouville's Theorem frequently on exam, and in problem sheets. More excitingly, Liouville's Theorem can prove the fundamental theorem of algebra.

# Hope you're excited as I am



# A theorem you've long used...

### Theorem (Fundamental Theorem of algebra)

*Let $p(z)$ be a non-constant polynomial with complex coefficients. Then there is a point $w \in \mathbb{C}$ such that $p(w) = 0$.*

### Proof.

Suppose not, and $p(z)$ has no roots. Then show $1/p(z)$ is bounded and analytic on $\mathbb{C}$, and apply Liouville's Theorem. $\square$

Using induction, it follows that a polynomial of degree $n$ has $n$ roots, counted with multiplicity.

### Note

The trick of dividing by a nonzero function appears frequently in applications.