Consequences of Cauchy's Theorem

Theorem (Cauchy's Theorem)

Suppose the function f is analytic on a simply connected region D. Then $\int_\gamma f dz = 0$ for all contours γ in D.

Corollary (Independence of Path)

Let f be analytic on a simply-connected region D and let γ_1 and γ_2 be any two paths in D from a to b. Then:

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$$

Proof.

If we do γ_1 then γ_2 backwards, we get a contour, and can apply Cauchy. □

Applications of path independence

Example (8.4)

Let γ be any path from $-i$ to i that cross $\mathbb{R}$ only between -1 and 1. Evaluate $\int_\gamma \frac{dz}{1-z^2}$.

Cauchy and ML: two great tastes that taste great together

Example (8.5)

Let α be any path in $D = \{z \in \mathbb{C} : |z| < 2\}$. Find B so that

$$\left| \int_\alpha \frac{\sinh(z)}{9 + e^z} dz \right| \leq B.$$

Application of Cauchy's Theorem: existence of primitives

I claimed that whenever Cauchy's Theorem applied, f has a primitive. More precisely:

Lemma

Let D be a simply connected domain, and let f be analytic on D. Let $p \in D$ be any point, and define a function F on D by $F(z) = \int_\alpha f(z)dz$ where α is any path in D from p to z. Then for any path $\gamma : [a, b] \to D$ we have

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a))$$

Furthermore, F is analytic on D, and $F' = f$.

Section 9: Cauchy's Integral Formula and Consequences

Theorem (Cauchy's integral formula)

Let γ be a simple contour described in the positive direction. Let w lie inside γ. Suppose that f is analytic on a simply connected region D containing γ and its interior. Then:

$$f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - w} dz.$$

Sketch of proof

1. Theorem 9.1 "Deforming contours": replace γ with $w + \varepsilon e^{2\pi i t}$ and take $\varepsilon \to 0$
2. $$???? \quad \frac{f(z) - f(w)}{z - w} \quad ????$$
3. ~~Profit~~ Prove the Theorem

Theorem 9.1 Deforming Contours

Theorem

Let γ be a simple contour described in the positive direction. Let z_0 be a point inside γ, and let C be another simple contour in positive direction, contained entirely inside γ. Suppose that f is analytic on a region D which contains γ, C and all points in between. Then

$$\int_\gamma f(z)dz = \int_C f(z)dz$$

- Crucially, D need not be simply connected
- Proof: join C and γ together by two paths, and rearrange contours to apply Cauchy.