

Section 9: Cauchy's Integral Formula and Consequences

Theorem (Cauchy's integral formula)

Let γ be a simple contour described in the positive direction. Let w lie inside γ . Suppose that f is analytic on a simply connected region D containing γ and its interior. Then:

$$f(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - w} dz.$$

Uses of Cauchy's Integral Formula

- ▶ Calculate contour integrals
- ▶ Values of f inside a contour determined by values on boundary
- ▶ Proving f has convergent Taylor series!

High level overview of Cauchy's Integral Formula

Many proofs we see have essential no real ideas to them – they're just unpacking definitions. Cauchy's integral formula has one real idea (Point 1) and one sneaky trick (Point 2).

Sketch of proof

1. Theorem 9.1 "Deforming contours": replace γ with $w + \varepsilon e^{2\pi it}$ and take $\varepsilon \rightarrow 0$

2.

$$???? \frac{f(z) - f(w)}{z - w} ????$$

3. Profit Prove the Theorem

Theorem 9.1 depends on Cauchy's Theorem, and it and the ideas that go into it are independently useful.

Theorem 9.1 Deforming Contours

Theorem

Let γ be a simple contour described in the positive direction. Let z_0 be a point inside γ , and let C be another simple contour in positive direction, contained entirely inside γ . Suppose that f is analytic on a region D which contains γ , C and all points in between. Then

$$\int_{\gamma} f(z) dz = \int_C f(z) dz$$

- ▶ Crucially, D need not be simply connected
- ▶ Proof: join C and γ together by two paths, and rearrange contours to apply Cauchy.

Proving Cauchy's Integral Formula

Let $C_\varepsilon(w)$ be the circle of radius ε around w . By Deforming Contours, we have

$$\int_\gamma \frac{f(z)}{z-w} dz = \int_{C_\varepsilon(w)} \frac{f(z)}{z-w} dz$$

Using the trick, we have that this is

$$= \int_{C_\varepsilon(w)} \frac{f(z) - f(w)}{z-w} dz + \int_{C_\varepsilon(w)} \frac{f(w)}{z-w} dz$$

We already computed that the second integral is $2\pi if(w)$, so we need to see the first integral tends toward zero.

Vanishing of the first integral

We use the ML estimate

- ▶ As $\varepsilon \rightarrow 0$, $z \rightarrow w$, and the integrand $\frac{f(z)-f(w)}{z-w} \rightarrow f'(w)$
- ▶ In particular, integrand is bounded by some M .
- ▶ The length of $C_\varepsilon(w) = 2\pi\varepsilon$.

Thus

$$\left| \int_{C_\varepsilon(w)} \frac{f(z) - f(w)}{z - w} dz \right| \leq M2\pi\varepsilon$$

And so the integral $\rightarrow 0$ as $\varepsilon \rightarrow 0$

Contour integrals via of Cauchy's Integral Formula

Example (Section 9.3)

Let γ be the simple, positively oriented triangular contour from 0 to $2 - 3i$ to $2 + 2i$ and back to zero. Evaluate

$$\int_{\gamma} \frac{e^z}{z-1} dz \quad \int_{\gamma} \frac{e^z}{z+1} dz \quad \int_{\gamma} \frac{1}{z^2-1} dz$$

$$\int_{\gamma} \frac{ze^{z^2}}{(z-1)(2z-1)} dz \quad \int_{\gamma} \frac{e^{z^2}}{z^2-1} dz$$

Tips and Tricks

- ▶ **Draw Picture** containing contour and "bad" points
- ▶ Avoid partial fractions – deform contour instead ☺