Classification of Singularities

By Laurent's Theorem, if $f(z)$ is analytic in a punctured disk around α, it has a convergent Laurent expansion

$$f(z) = \sum_{n \in \mathbb{Z}} a_n (z - \alpha)^n$$

Three possibilities:

Removable singularity: None of the a_n with $n < 0$ are nonzero

A pole: Only finitely many a_n with $n < 0$ are nonzero

Essential singularity: Infinitely many a_n with $n < 0$ are nonzero

Typical Question (first step to using Residue Theorem):

Find and classify the singularities of $f(z)$, and find the residue at each one.

Examples:

$$\frac{1}{1+z^2} \qquad \frac{1}{e^z-1} \qquad z\cos\left(\frac{1}{z-1}\right)$$

$$\frac{\cos(z)}{z^2\sin(z)} \qquad \frac{\tan z}{z}$$

Easy (and examinable!) theorems about poles

Theorem

f has a pole of order k at α if and only if

$$f(z) = \frac{g(z)}{(z-\alpha)^k}$$

where $g(z)$ analytic and nonzero in some disk around α.

Theorem

If f has a zero of order k at α, then $1/f$ has a pole of order k at α.

Corollary

If f has a zero of order m at α, and g has a zero of order n at α, then

- $\frac{f}{g}$ *has a pole of order $n-m$ if $n > m$*
- $\frac{f}{g}$ *has a removable singularity if $m \geq n$*

Easy way to find residues at poles

Theorem

Suppose that f has a pole of order k at α. Then

$$\mathrm{Res}\{f;\alpha\} = \frac{1}{(k-1)!}\lim_{z\to\alpha}\frac{d^{k-1}}{dz^{k-1}}(z-\alpha)^k f(z)$$

Proof.

Just compute the right hand side. □

Corollary

If $f = g/h$, where g and h are analytic at α,
$g(\alpha) \neq 0, h(\alpha) = 0, h'(\alpha) \neq 0$, then f has a simple pole at α and

$$\mathrm{Res}\{f;\alpha\} = \frac{g(\alpha)}{h'(\alpha)}$$