Announcements

☹ Strike ☹

- (First wave?) Next Week, December 1,2,3
- GLobal Issues: Pensions, Four Fights
- Local Issues: Archaeology, Modern Languages

Other items

- Can still hand in second homework for feedback
- Exam still being finalized, more information soon
- Finish notes next week! Revision topics?
- Will have office hours + revision sessions in January

Home stretch:

Preparing for the Residue Theorem

Laurent Series: First example

For $|z| < 1$, we have

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

As $|z| \to \infty, \frac{1}{1-z} \to 0$. Can we analyze how?

Yes!

As $|z| \to \infty, |1/z| \to 0$, and in particular $|1/z| < 1 \ldots$

Laurent Series: Definition

A Laurent series about a is a like a Taylor Series but where we also allow negative powers.

Definition

A Laurent series around a is a series of the form

$$\sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

If the series converges to a function $f(z)$ for $0 < |z-a| < r$, the coefficient a_{-1} is called the *residue of f at a*.

Generally will converge on some (perhaps empty) annulus $r < |z-a| < R$, and give a holomorphic function there.

Why?

- Replacement for Taylor series around isolated singularities
- Easy to integrate over a contour around a

Classification of Singularities

By Laurent's Theorem, if $f(z)$ is analytic in a punctured disk around α, it has a convergent Laurent expansion

$$f(z) = \sum_{n \in \mathbb{Z}} a_n (z - \alpha)^n$$

Three possibilities:

Removable singularity None of the a_n with $n < 0$ are nonzero

A pole Only finitely many a_n with $n < 0$ are nonzero

Essential singularity Infinitely many a_n with $n < 0$ are nonzero

Punchline first:

Only for essential singularities will you need to compute the Laurent series to find the pole!

Removable singularities: no negative powers

f has a removable singularity at α means it has no negative powers of $z - \alpha$ in its Laurent series.
But then it's Laurent series is really a Taylor series, and it makes sense to plug in α. Thus f extends to an analytic function around α.

Examples:

1. $\frac{z^2-1}{z-1}$
2. $\frac{\sin(z)}{z}$

Since the Laurent series has no negative powers, the residue of a removable singularity is always zero.

Poles: only finitely many negative terms

Definition

We say that $f(z)$ has a *pole of order* k at α if its Laurent series $f(z) = \sum_{n \in \mathbb{Z}} a_n (z - \alpha)^n$ has $a_{-k} \neq 0$, but $a_n = 0$ for $n < -k$.

In other words

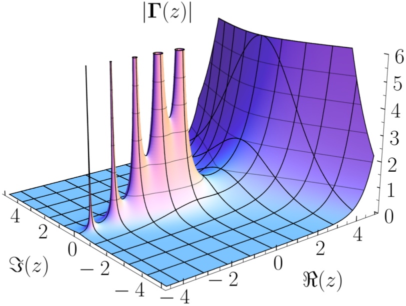
$$f(z) = \sum_{n \geq -k} a_n (z - \alpha)^n$$

A pole of order one is also called a *simple pole*.

Examples: poles of order 2

1. $\frac{e^z}{z^2}$
2. $\tan^2(z)$

Poles of the gamma function



$\Gamma(z)$ extends $(n-1)!$ to an analytic function, and has simple poles at the non-positive integers

Essential singularity: infinitely many negative powers

Definition

If the Laurent series of $f(z)$ around α has infinitely many $a_k \neq 0$ with $k < 0$, then we say α is an *essential singularity* of f

Examples

- $e^{1/z}$
- $\cosh(1/z)$

Theorem (Great Picard's Theorem – just for culture)

If $f(z)$ has an essential singularity at α, then on any punctured disk around α, $f(z)$ takes on all possible complex values, with at most one exception, infinitely often.

So $\lim_{z\to\alpha} |f(z)| = \infty$ for a pole, but horribly doesn't exist for an essential singularity.