"ML-bounds": M=Maximum, L=Length

Theorem

Suppose that f is continuous on a path γ of length L, and that $|f(z)| \leq M$ on γ. Then $|\int_\gamma f dz| \leq ML$

Proof.

$$\left|\int_\gamma f(z)dz\right| = \left|\int_a^b f(z(t))z'(t)dt\right|$$
$$\leq \int_a^b |f(z(t))||z'(t)|dt$$
$$\leq \int_a^b M|z'(t)|dt = ML$$

$\square$

The second line is triangle inequality for integrals.

Applications of the ML-inequality

Why do ML-inequalities?

The ML-inequality will help prove two of our big theorems.
We will take limits of paths where M or L are going to zero, and conclude the integral goes to zero.

"Toy" uses appear on exam

Example (From notes)

Let γ be a line segment lying with $D = \{z \in \mathbb{C} : |z| < 1$. Give an upper bound for

$$\int_\gamma \left(\frac{\operatorname{Re} z + z^2}{3 + \overline{z}} \right) dz$$

Section 8: Cauchy's Theorem

Theorem (Cauchy's Theorem)

Suppose the function f is analytic on a simply connected region D. Then $\int_\gamma f dz = 0$ for all contours γ in D.

Example

$$\int_{C_{0,1/2}} \frac{\sin(e^{3z}\cos(z))}{1+z^3} dz = 0$$

Example (Necessary to ask that D is simply connected)

The function $1/z$ is analytic on $\mathbb{C} \setminus \{0\}$. Letting $\gamma = e^{it}, 0 \leq t \leq 2\pi$ we have

$$\int_\gamma \frac{1}{z} dz = 2\pi i$$

No Cauchy, because $\mathbb{C} \setminus \{0\}$ isn't simply connected.

The proof of Cauchy's Theorem

The proof of Cauchy's Theorem is hard

Because the hypotheses are so weak!

- We won't prove it
- Notes: Green's Theorem + Cauchy-Riemann would prove it
- But Green's theorem requires *continuous* partial derivatives; we only have that f has a derivative!

Proof *is* in the recommended textbooks...

Theorem (Green)

Let Δ be the region bounded by an anticlockwise contuous γ, and let f and g have continous partial derivatives. Then:

$$\int_\gamma f\,dx + g\,dy = \iint_\Delta \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\,dx\,dy$$

A non-continuous derivative

Here's an example that shows that for *real* functions, the derivative of a function can be non-continuous:

Example

Let

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Then

$$f'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

exists at 0 but is not continuous there.