Sequences in $\mathbb{C}$ – just as in $\mathbb{R}$

Definition

Let a_n be a sequence of complex numbers. Then we say the sequence converges to L, and write

$$\lim_{n\to\infty} a_n = L$$

if for all $\varepsilon > 0$ there exists an N so for $n > N$ we have $|a_n - L| < \varepsilon$

Lemma

$$\lim_{n\to\infty} a_n = L \iff \begin{array}{l} \lim_{n\to\infty} \mathrm{Re}(a_n) = \mathrm{Re}(L) \\ \lim_{n\to\infty} \mathrm{Im}(a_n) = \mathrm{Im}(L) \end{array}$$

Proof.

$$\max\left(|\,\mathrm{Re}(z-L)|, |\,\mathrm{Im}(z-L)|\right) \leq |z-L| \leq |\,\mathrm{Im}(z-L)| + |\,\mathrm{Re}(z-L)|$$

$\square$

Series in $\mathbb{C}$ – just as in $\mathbb{R}$

Definition

Let a_n be a sequence of complex numbers. Then we say the series $\sum_{i=0}^{\infty} a_i$ converges to L if the sequence of partial sums $S_n = \sum_{i=0}^{n} a_i$ converges to L.

Definition

A series $\sum a_n$ is *absolutely convergent* if $\sum |a_n|$ converges.

Two tools:

- Comparison test
- Geometric series

We'll mostly be interested in power series

Definition

Suppose that $a_n, z_0 \in \mathbb{C}$. A series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is called a *power series centred at* z_0.

We will be interested in what values of z a given power series converges; for those values, we will have a function of z. E.g.

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \forall z \in \mathbb{C}$$

Radius of convergence

Theorem

Suppose $w \neq 0$ and $\sum a_n w^n$ converges. Then $\sum a_n z^n$ is absolutely convergent for all z with $|z| < |w|$

Theorem (Abel)

For any power series, either:

1. *The power series converges only at $z = 0$.*
2. *The power series is absolutely convergent for all $z \in \mathbb{C}$*
3. *There is a real number R so that the power series is absolutely convergent if $|z| < R$, and divergent if $|z| > R$*

R is called the radius of convergence. It is zero in case one, and infinite in case 2.

The ratio test

The radius of convergence always exists (though it may be 0 or ∞), but not clear how to find. We will mostly use:

Theorem (Ratio Test)

If

$$\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = R$$

Then R is the radius of convergence.

Proof.

Idea: use comparison test to compare to geometric series $\square$

- Can't apply Ratio Test to "most" power series.
- *Can* apply Ratio Test to most power series we'll see!

Examples

Find the radius of convergence of the following functions.

$$\sum_{n=0}^{\infty} (\sinh(n)) z^n \tag{1}$$

$$\sum_{n=1}^{\infty} \frac{(2i)^n z^n}{n} \tag{2}$$

$$\sum_{n=1}^{\infty} \frac{(2i)^n z^{3n}}{n} \tag{3}$$

$$\sum_{n=0}^{\infty} \frac{(2n)!n!}{(3n)!} z^n \tag{4}$$

Convergent Power series give analytic functions

Define $f(z) = \sum a_n z^n$ inside the radius of convergence. Is $f(z)$ analytic? We'd like to argue:

$$\begin{aligned} f'(z) &= \frac{d}{dz} \sum a_n z^n \\ &= \sum \frac{d}{dz} a_n z^n \\ &= \sum n a_n z^{n-1} \end{aligned}$$

We're being Evil Kermit

- Not clear we can move derivative inside sum
- Not clear final power series converges

Hence: Power series give analytic functions!

Will see later this gives *all* analytic functions!