He's *not* a real smooth function

The following function from $\mathbb{R} \to \mathbb{R}$ is infinitely differentiable everywhere, but the Taylor series at 0 converges nowhere:

$$f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-1/x} & x > 0 \end{cases}$$

Analytic Functions have Taylor Series

Theorem

Let f be analytic on $\Delta = \{z : |z - z_0| < r\}$, where $r > 0$. Then f has a Taylor expansion about z_0 that is valid on all of Δ:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

Proof.

CIF + Expand $1/(w - z)$ as geometric series in $(z - z_0)/(w - z_0)$
Warning: Usual role of w and z in CIF reversed for this proof. □

Cool corollary

The radius of convergence around z_0 is the distance to the first point w where f isn't analytic.
In particular, if f is entire (analytic on all of $\mathbb{C}$), then its Taylor expansions converge everywhere!

Calculating Taylor series

When possible, avoid taking derivatives

Too much work.

Instead...

- Build from Taylor series you know: $1/(1-z), \exp, \sin, \ldots$
- Helps psychologically to substitute $w = z - z_0$
- Can differentiate / integrate Taylor series term by term

Examples: find Taylor series of...

1. $1/(z+1)$ around $z = 2$
2. $z^3 \cosh(z^2)$ around $z = 0$
3. $1/(1-z)^2$ around $z = 0$

Zeroes of functions

Definition

Let f be analytic on a region D. We say $w \in D$ is a *zero of f* if $f(w) = 0$. We say w is a *zero of order k* if $f(w) = f'(w) = \cdots = f^{(k-1)}(w) = 0$, but $f^{(k)}(w) \neq 0$.

Example

$z \sin(z)$ has a zero of order 2 at 0, and a zero of order 1 at $k\pi$ for $0 \neq k \in \mathbb{Z}$

Lemma

$f(z)$ has a zero of order k at a if and only if $f(z) = (z-a)^k g(z)$, where $g(z)$ is analytic on nonzero on an open set containing a.

Corollary

If $f(z)$ has a zero of order m at a, and $g(z)$ has a zero of order n at a, then $f \cdot g$ has a zero of order $m + n$ at k.