

Analytic Functions have Taylor Series

Theorem

Let f be analytic on $\Delta = \{z : |z - z_0| < r\}$, where $r > 0$. Then f has a Taylor expansion about z_0 that is valid on all of Δ :

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

Proof.

CIF + Expand $1/(w - z)$ as geometric series in $(z - z_0)/(w - z_0)$

Warning: Usual role of w and z in CIF reversed for this proof. \square

Cool corollary

The radius of convergence around z_0 is the distance to the first point w where f isn't analytic.

In particular, if f is entire (analytic on all of \mathbb{C}), then its Taylor expansions converge everywhere!

Calculating Taylor series

When possible, avoid taking derivatives

Too much work.

Instead...

- ▶ Build from Taylor series you know: $1/(1 - z)$, \exp , \sin , ...
- ▶ Helps psychologically to substitute $w = z - z_0$
- ▶ Can differentiate / integrate Taylor series term by term

Examples: find Taylor series of...

1. $1/(z + 1)$ around $z = 2$
2. $z^3 \cosh(z^2)$ around $z = 0$
3. $1/(1 - z)^2$ around $z = 0$

Zeroes of functions

Definition

Let f be analytic on a region D . We say $w \in D$ is a zero of f if $f(w) = 0$. We say w is a zero of order k if $f(w) = f'(w) = \dots = f^{(k-1)}(w) = 0$, but $f^{(k)}(w) \neq 0$.

Example

$z \sin(z)$ has a zero of order 2 at 0, and a zero of order 1 at $k\pi$ for $0 \neq k \in \mathbb{Z}$

Lemma

$f(z)$ has a zero of order k at a if and only if $f(z) = (z - a)^k g(z)$, where $g(z)$ is analytic and nonzero on an open set containing a .

Corollary

If $f(z)$ has a zero of order m at a , and $g(z)$ has a zero of order n at a , then $f \cdot g$ has a zero of order $m + n$ at a .