Today: Consequences of the Cauchy-Riemann Equations

We ended last time by proving:

Theorem (Cauchy-Riemann Equations)

For $z = x + iy$, write $f(z) = u(x, y) + iv(x, y)$ with u, v real.
Then if f is differentiable at z_0, we have:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{at } z_0$$

Today: what does this tell us about f?

1. Silly, but easy to examine: If u, v are related in any other way, f is *highly* constrained
2. Important, but not traditionally examined: f is a *conformal mapping*
3. Important, examined: The real and imaginary parts of f are *harmonic*

Section 5.8: When u and v are related

- Cauchy-Riemann relates between $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$
- If we have more relations, then f is *very* constrained

Example (Similar to those in notes)

Suppose that f is differentiable on a connected domain, and that its real and imaginary parts satisfy $u = v^2$. Prove that f is constant.

Holomorphic functions are conformal maps

In MAS211 you looked at the derivative of a map $f : \mathbb{R}^n \to \mathbb{R}^m$ as a linear map $Df : \mathbb{R}^n \to \mathbb{R}^m$, and hence as a matrix. The entries are the partial derivatives, so for $f : \mathbb{C} \to \mathbb{C}$

$$Df = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

If $f : \mathbb{C} \to \mathbb{C}$ is differentiable at z_0, this linear map corresponds to multiplication by the complex number $f'(z_0) = a + bi$. The Cauchy-Riemann equations just enforce this:

$$Df(z_0) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Hence the derivative is a rotation + a scaling, and *preserves angles*. Such a map from $\mathbb{R}^2 \to \mathbb{R}^2$ is called *conformal*.

Motivation for harmonic functions: important PDEs

The Laplacian operator, written ∇^2 or Δ, acts on functions $g : \mathbb{R}^2 \to \mathbb{R}$ by

$$\nabla^2 g = \nabla \cdot \nabla g = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}$$

and occurs in many PDEs important in applied math.

Examples

Let $f(x, y, t)$ be a function of two space variables and one time variable.

- The heat equation $\frac{\partial f}{\partial t} = \nabla^2 f$
- The wave equation $\frac{\partial^2 f}{\partial t^2} = \nabla^2 f$

A steady state solution to either of these equations would be $\nabla^2 f = 0$.

Harmonic Functions

Definition

A function $u : \mathbb{R}^2 \to \mathbb{R}$ is *harmonic* if $\nabla^2 f = 0$

Lemma

Let $f(z) = u(x, y) + iv(x, y)$ be analytic on a domain D. Then u and v are harmonic on D

Proof.

Cauchy-Riemann equations + mixed partials are equal. □

This gives us lots of harmonic functions.

Does this give us *all* harmonic functions?

Given a harmonic function $u(x, y)$ on a domain, is it the real part of an analytic function $f(z)$?

Yes, on a *simply-connected* domain.

When is u are the real part of analytic functions?

From the 2012-2013 exam

(iii) Find all the functions k analytic on $\mathbb{C}$ with $\mathrm{Re}\,(k(x+iy)) = 2x - \sinh x \sin y$, giving an explicit expression for $k(z)$ in terms of z. Show that you have found **all** the functions satisfying the above conditions. ***(6 marks)***

The real part of an analytic function is harmonic

First, check $\nabla^2 u = 0$. If not, the answer is no.

If it is, find f' using Cauchy-Riemann

$$f' = \frac{\partial}{\partial x}\Big(u(x,y) + iv(x,y)\Big) = u_x + iv_x = u_x - iu_y$$

This gives us f' in terms of x and y. We'd *like* to write f' in terms of z, and integrate to find f. But how?

Maybe we need a clever little trick....

Dr. Hart's "Clever little trick"

Given:
We know f' in terms of x and y, want it terms of z.

Guess:
Set $y = 0$; to get $f'(x)$ in terms of just x. Integrate to get $f(x)$. Guess that this is actually formula for $f(z)$

Check:
Show that $\text{Re}(f) = u(x, y)$

To find *all* such k:

Lemma
Suppose that f and g are analytic on a region D and that $\text{Re}(f) = \text{Re}(g)$ on D. Then $f = g + ia$ for some $a \in \mathbb{R}$.