Complex Analysis: What's all this, then?

Up until now: derivatives and integrals of functions $f : \mathbb{R}^n \to \mathbb{R}^m$

Derivatives and integrals of $f : \mathbb{C} \to \mathbb{C}$

- Easier than real case!
- Analysis not a prerequisite – not focused on *doing* proofs
- *Using* theorems correctly, building on each other

Theorem (Fundamental Theorem of Algebra)

Let $p(z) \in \mathbb{C}[x]$ be a nonconstant polynomial. Then there is a $w \in \mathbb{C}$ with $p(w) = 0$.

The Residue Theorem has bonkers applications:

$$\int_{-\infty}^{\infty} \frac{\cos(\pi x)}{(1+x^2)^2}\, dx = \frac{\pi}{2} e^{-\pi}(\pi + 1)$$

Calculating Fourier and Laplace transforms. Number Theory.

Written Resources

Main resource: Dr. Hart's notes

- Fine-tuned toward the exam
- Many worked examples

Recommended texts: extra material + viewpoints, more rigour

- Beck et al (free and legal online!)
- Priestley (pdf through library)
- Stewart and Tall (old edition is cheap online)

Slides

- Posted online before lecture; space to write notes
- Roughly follow Dr. Hart's notes

Assessment and feedback

100% of marks from final exam

- Exam questions generally very similar to past years
- All questions mandatory, probably 4
- Forthcoming exam page will have more detail

Exercises and feedback

- Exercise and hints sheets on Blackboard DO DURING TERM
- Will collect a few to mark for feedback
- Worked solutions to all exercises will go up shortly after that

Interactive “clicker questions” in lecture

- Using TurningPoint app/webpage
- Everyone gets instant feedback on how things are going

Review of complex numbers

The complex plane $\mathbb{C} = \{z : z = x + iy; x, y \in \mathbb{R}\}$.

- $x = \mathrm{Re}(z)$ is called the *real part*
- $y = \mathrm{Im}(z)$ is called the *imaginary part*
- $\bar{z} = x - iy$ is called the *complex conjugate*
- $|z| = \sqrt{x^2 + y^2}$ is called the *modulus* of z

Complex numbers form a field

Can add, multiple, subtract by any complex number, and divide by nonzero complex numbers.

A few tricks

$\mathbb{C} \cong \mathbb{R}^2$

An equation $z = w$ of complex numbers can be viewed as a pair of equations between real numbers:

$$z = w \iff \mathrm{Re}(z) = \mathrm{Re}(w) \text{ and } \mathrm{Im}(z) = \mathrm{Im}(w)$$

Why care about $\overline{z}$?

$$|z|^2 = z \cdot \overline{z}$$

This lets us find z^{-1}.

Example

What's $\frac{1-i}{3+4i}$?

Geometry of complex numbers

Theorem (Euler)

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Polar form / "modulus-argument" form

If we write $z = x + iy$ in polar coordinates:

$$z = r\cos(\theta) + ir\sin(\theta) = re^{i\theta}$$

Multiplying and dividing easier in polar form!

$$(re^{i\theta})(se^{i\phi}) = (rs)e^{i(\theta+\phi)}$$

$$\frac{re^{i\theta}}{se^{i\phi}} = (r/s)e^{i(\theta-\phi)}$$

Powers and roots

Theorem (De Moivre)

Let $\theta \in \mathbb{R}$ and $n \in \mathbb{Z}$. Then

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

Proof.

Apply Euler's theorem to both sides of $(e^{i\theta})^n = e^{in\theta}$, or prove directly for $n \in \mathbb{N}$ using induction and trig formulas. □

Theorem

A complex number $z = re^{i\theta}$ has precisely n nth roots:

$$\sqrt[n]{r} e^{i(\theta + 2k\pi)/n} \qquad k \in \{0, 1, \ldots, n-1\}$$