Actually computing line integrals

$$\int_\gamma f(z)\mathrm{d}z := \int_a^b f(z(t))z'(t)\mathrm{d}t$$

Example

Let γ be the path that's a straight line from 0 to 1, and then a straight line from 1 to $2+i$. Compute:

$$\int_\gamma \mathrm{Re}(z)\mathrm{d}z$$

$$\int_\gamma \mathrm{Im}(z)\mathrm{d}z$$

$$\int_\gamma z\mathrm{d}z$$

The most important example

Recall: $C_r(a)$ is the anti-clockwise circle of radius r around a.

An mysterious computation:

Let $n \in \mathbb{Z}$

$$\int_{C_r(a)} \frac{1}{(z-a)^n} \mathrm{d}z = \begin{cases} 0 & n \neq 1 \\ 2\pi i & n = 1 \end{cases}$$

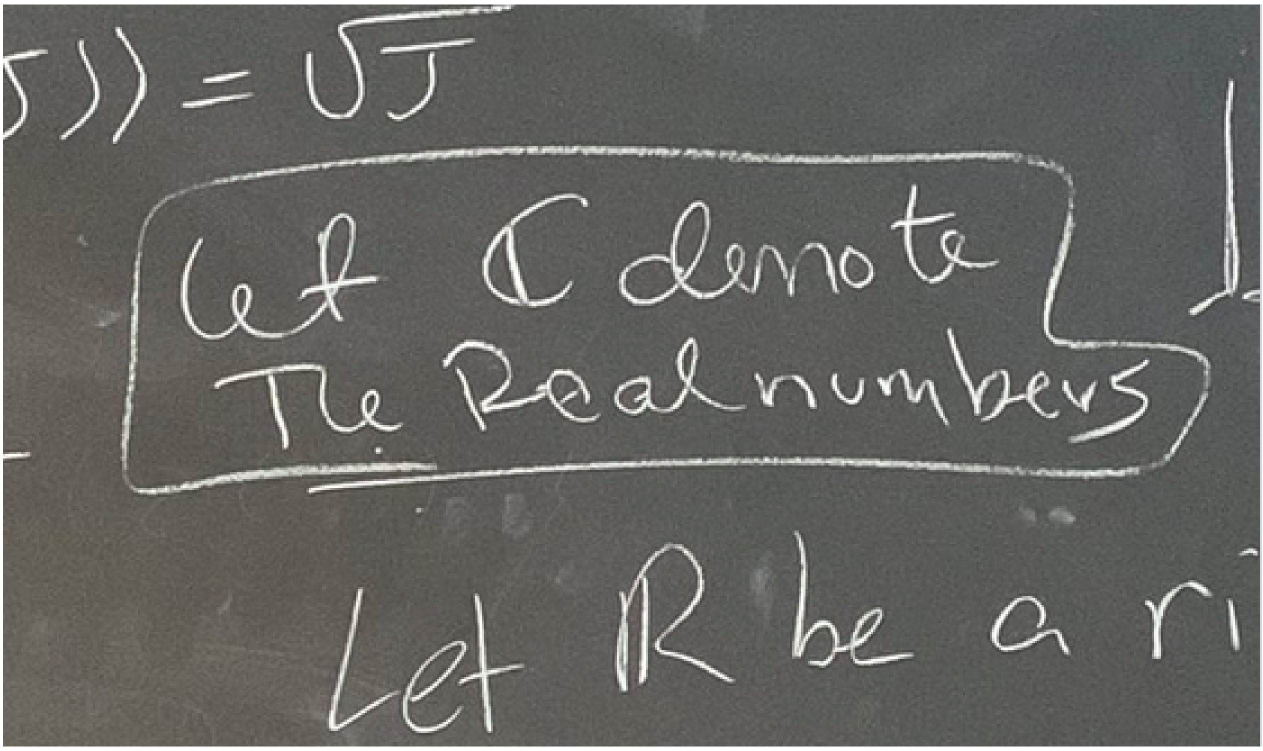
Independent of a and r, works for n negative, too!

Coming attractions – conceptual explanation!

- *Antiderivatives* explain why the answer is zero unless $n = 1$
- *Cauchy's theorem* explains why it's independent of r
- *Residue theorem* reduces *any* integral to this computation!

Clicker Session
Turning Point app or
ttpoll.eu

Seen on twitter – “Chaotic evil” maths



It's not "wrong", but it is painful.
Some definitions in the notes are similar...

Types of Sets – a topology taster

The following Definition in the notes is NOT STANDARD USAGE. I won't use it.

Definition (4.1 in notes)

A *neighbourhood* of $z_0 \in \mathbb{C}$ is an open disc about z_0, i.e., it's of the form

$$\{z \in \mathbb{C} : |z - z_0| < \delta\}$$

for some $\delta > 0$.

Definition (4.2 in notes. This definition is standard)

A set $D \subseteq \mathbb{C}$ is said to be *open* if for each point $z_0 \in D$ there's an open disc contained in D and containing z_0.

Easier to think in terms of pictures...

“Can’t be cut in two” vs. “Can get from place to place”

Two possible intuitions behind being connected.

Definition (Connected)

A subset $X \subseteq \mathbb{C}$ is *connected* if we can’t find two nonempty open sets $U, V \subset \mathbb{C}$ with $U \cap V = \emptyset$ and $X \subset U \cup V$.

Intuition: If $X \subset U \cap V$, then $X \cap U, X \cap V$ cuts X into two pieces.

Definition (Path-connected)

A subset $X \subseteq \mathbb{C}$ is called *path-connected* if for any two points $x, y \in X$ we can find a path $\gamma \in X$ with initial point x and final point y

- In general, path-connected $\implies$ connected, but not converse
- If X is open, X is connected $\iff$ X is path-connected.

A little bit more

Many of our Theorems are going to hold for nice open sets, so we develop shorthand for recording this.

Definition (4.4 in notes)

A non-empty, open, connected set is called a *region*.

Definition (4.5 in notes)

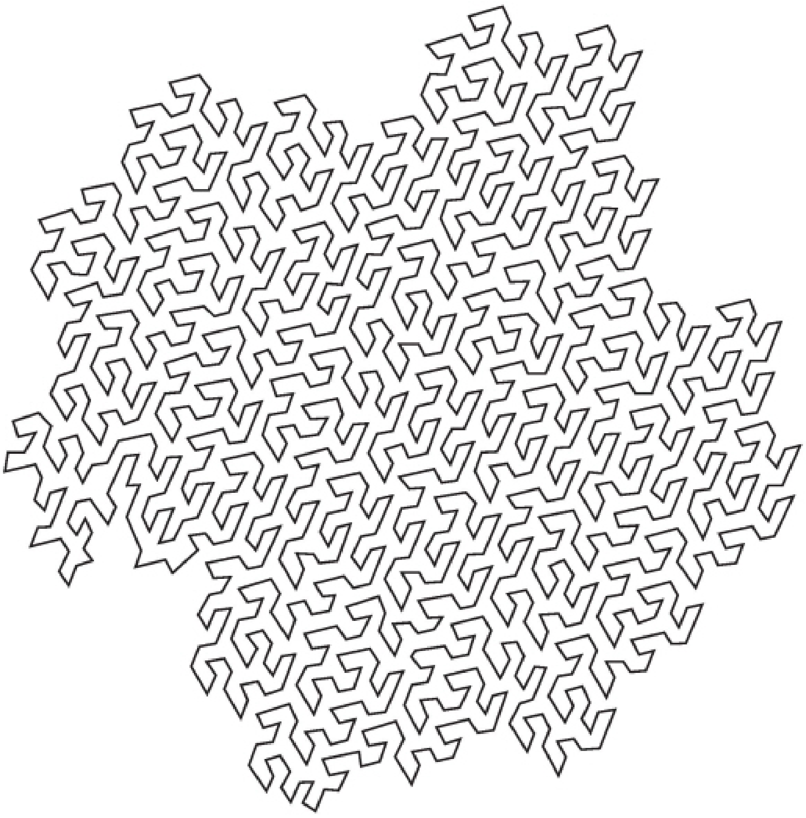
A region D is said to be *simply connected* if it has no 'holes'; i.e., if every point in the interior of any simple contour in D is contained in D.

Mathematical culture:

We're implicitly using the Jordan Curve theorem – that a simple curve closed in the plane has an inside and an outside.

Why is this hard?

Pick a point in the middle – is it inside, or outside?



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