Ended last time computing line integrals

We finished one basic example.

Do we want another easy example?

A mysterious (but important!) example

Let $n \in \mathbb{Z}$

$$\int_{C_r(a)} \frac{1}{(z-a)^n} \mathrm{d}z = \begin{cases} 0 & n \neq 1 \\ 2\pi i & n = 1 \end{cases}$$

Independent of a and r, works for n negative, too!

Will revisit throughout module:

$$\int_{C_r(a)} \frac{1}{(z-a)^n} \mathrm{d}z = \begin{cases} 0 & n \neq 1 \\ 2\pi i & n = 1 \end{cases}$$

Coming attractions – conceptual explanation!

- *Antiderivatives* explain why the answer is zero unless $n = 1$
- *Cauchy's theorem* explains why it's independent of r
- *Residue theorem* reduces *any* integral to this computation!

Section 5: Derivatives. First: limits, continuity

Definition (Limits)

Let f be defined on some punctured neighborhood of z_0. Then we say

$$\lim_{z \to z_0} f(x) = a$$

if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |z - z_0| < \delta$, then $|f(z) - f(z_0)| < \varepsilon$.

Definition (Continuous)

Let f be defined in a neighborhood of a. We say f is continuous at a if $\lim_{z \to a} f(z) = f(a)$.

These are identical to limit/continuity for $f : \mathbb{R}^2 \to \mathbb{R}^2$

Doesn't use the multiplicative structure of $\mathbb{C}$.

The definition of the derivative *looks* the same

Definition (The derivative)

Let $f : \mathbb{C} \to \mathbb{C}$ be defined on a neighbourhood of z_0. The derivative of f at z_0, if it exists, is

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Could also use the definition with $h \to 0$, but now $h \in \mathbb{C}$.

Many familiar things follow:

- $\frac{d}{dz} z^n = nz^{n-1}$
- $\frac{d}{dz} e^z = e^z$
- The derivative is linear
- Chain rule, product rule, quotient rule
- ...

BUT some very nice functions *aren't* differentiable

Example

Let $f(z) = \text{Re}(z)$. Then f is not differentiable at any point in $\mathbb{C}$.