Computing real integrals with the Residue Theorem

Basic idea is very flexible:

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R,S\to\infty} \int_{-R}^{S} f(x)dx$$

- Include the finite integral as part of a contour integral
- Calculate the contour integral using residue theorem
- As $R, S \to \infty$ contributions of other parts of contour $\to 0$

ML Estimates often work for last point

Basic example / sanity check:

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}dx = \pi$$

Specific formulation appearing in notes on exam

Older exams wanted you to memorize this

Theorem

Let $f(z) = \frac{p(z)}{q(z)} e^{i\lambda z}$ where $\lambda \in \mathbb{R}, \lambda > 0$, $p(z), q(z)$ are polynomials with $\deg q > \deg p$, no common zeroes, and q has no real roots. Then

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} e^{i\lambda x} dx = \sum_{i=1}^{k} \mathrm{Res}\{f; z_k\}$$

where $z_1, z_2, \dots, z_k$ are the zeros of q in the upper half plane.

Proof.

Follow plan from last slide; if $\deg(q) + 1 < \deg(p)$ can just use ML-estimates, otherwise we need to sweat a bit more. $\square$

Applying our specific formulation to compute real integrals

Take real and imaginary parts of our integrand

If p, q have real coefficients, and x is real, then

$$\operatorname{Re} \frac{p(x)}{q(x)} e^{i\lambda x} = \frac{p(x)}{q(x)} \cos(\lambda x) \quad \operatorname{Im} \frac{p(x)}{q(x)} e^{i\lambda x} = \frac{p(x)}{q(x)} \sin(\lambda x)$$

Examples from Section 12.3 of Notes

$$\int_{-\infty}^{\infty} \frac{x \sin(\pi x)}{x^2 + 2x + 5} dx = -\pi e^{-2\pi}$$

$$\int_{0}^{\infty} \frac{\cos(\pi x)}{(1 + x^2)^2} dx = \frac{\pi(\pi + 1)e^{-\pi}}{4}$$

$$\int_{0}^{\infty} \frac{\cos^2(x)}{1 + x^2} dx = \frac{\pi(1 + e^2)}{4e^2}$$