MAS341 Graph Theory
2014 exam solutions

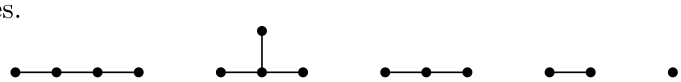
Question 1

(i)(a) A tree is a connected graph with no cycles. If it has n vertices then it has $n-1$ edges.

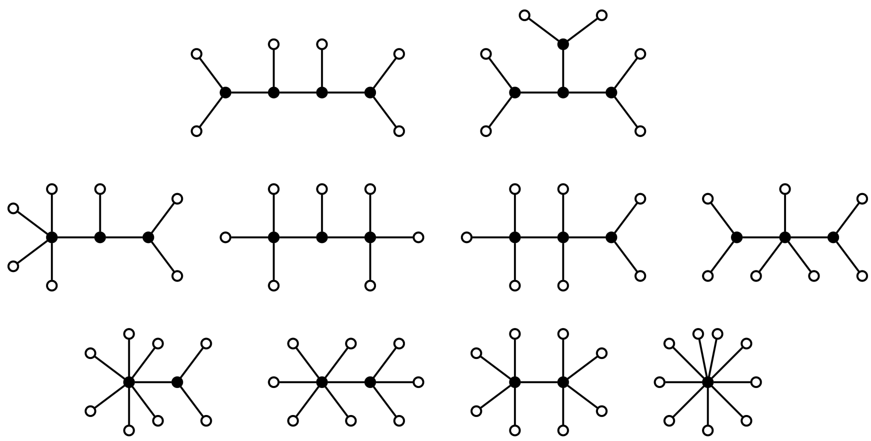
(i)(b) Let T be a tree with $n > 1$ vertices. If it has at most one leaf then the smallest degree is at least 1 and all other degrees are at least 2, so the sum of the degrees is at least $1 + 2(n-1) = 2n-1$. But by handshaking, the sum of degrees is $2e(T) = 2n-2$, which is a contradiction.

(i)(c) Let T be a tree with $n > 1$ vertices, none of degree 2, and l leaves. Every vertex which is not a leaf has degree at least 3, so the sum of the degrees is at least $l + 3(n-l) = 3n-2l$. By handshaking, the sum of degrees is $2e(T) = 2n-2$, and so $2n-2 \geq 3n-2l$, i.e. $l \geq (n+2)/2$.

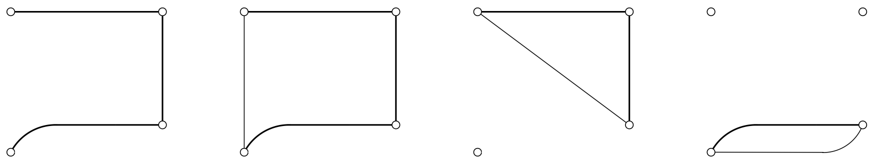
(i)(d) By (c), such a tree has at least 6 leaves. If these are removed, the remaining vertices will form a tree on at most 4 vertices, and there are five possibilities.



Then we must add leaves to these with each other vertex having degree at least 3. There are ten ways to do this.



(ii)(a) A spanning tree (bold) and the three fundamental cycles are shown below.



(ii)(b) From Kirchhoff's first law at three vertices we get

$$i_0 = i_1 \tag{1}$$
$$i_1 = i_2 + i_3 + i_4 \tag{2}$$
$$i_0 = i_4 + i_5 \tag{3}$$

From Kirchhoff's second law on the fundamental cycles we get

$$76 = 2i_1 + 6i_4 \tag{4}$$
$$2i_2 = 3i_3 \tag{5}$$
$$i_0 = i_4 + i_5 \tag{6}$$

From (1) and (2) we get

$$i_0 = i_2 + i_3 + i_4 \tag{7}$$

and combining (7) with (3) and (5) gives

$$i_5 = i_2 + i_3 = \tfrac{5}{3} i_2 \tag{8}$$

From (4) and (6), then using (8),

$$6i_4 = 2i_2 + 3i_5 = 7i_2$$

Consequently,

$$i_1 = i_2 + i_3 + i_4 = i_2(1 + \tfrac{2}{3} + \tfrac{7}{6}) = \tfrac{17}{6} i_2$$

and

$$76 = 2i_1 + 6i_4 = \tfrac{38}{3} i_2$$

so $i_2 = 6$A. Now $i_3 = 4$A, $i_4 = 7$A, $i_5 = 10$A, $i_0 = i_1 = 17$A.

Question 2

(i) A graph is Eulerian if there is a closed trail using every edge, semi-Eulerian if there is a trail using every edge which is not closed, Hamiltonian if there is a cycle using every vertex, and semi-Hamiltonian if it is not Hamiltonian but there is a path using every vertex. The graph K_4 is Hamiltonian but not Eulerian.

(ii) We prove this by induction on the number of edges; if there are no edges we are done. If G has some edges, let H be a component with at least one edge. Every vertex in H must have degree at least 2, so $2e(H) \geq 2|H|$, so H is not a tree. Consequently H contains a cycle, C. Remove the edges of C from G to obtain G'. G' has fewer edges than G, and the degree of every vertex has changed by 0 or 2 so it still has all degrees even. So by the induction hypothesis, the edges of G' may be partitioned into cycles, and adding C to such a partition gives a partition of G.
G need not be Eulerian, since it does not have to be connected.

(iii) Draw a graph G with a vertex for each player and an edge between two players for every game they play. There are no loops, so if everyone plays an even number of games then all degrees are even. Partition the edges into cycles. For each cycle, go round the cycle in one direction and arrange the colours of those games so that each edge goes from the white player to the black player. Each player plays white once and black once in each cycle (s)he is in, so plays the same number of each overall.
It is not always true that we can arrange the colours so that no player plays the same number of each. If there are three players, each pair of whom play once, then at most one player can play black in both his/her games, and at most one can play white in both, so at least one must play black and white once each.

(iv) The degree sum is $2e(G)$, which is even, so there are an even number of vertices of odd degree. Divide the k players into $k/2$ pairs, and add one game between each of these pairs. The resulting graph has all degrees even. Now assign colours as above so that every player plays white and black the same number of times. Then remove the $k/2$ extra games. We remove at most one game from each player, so the difference is at most 1.

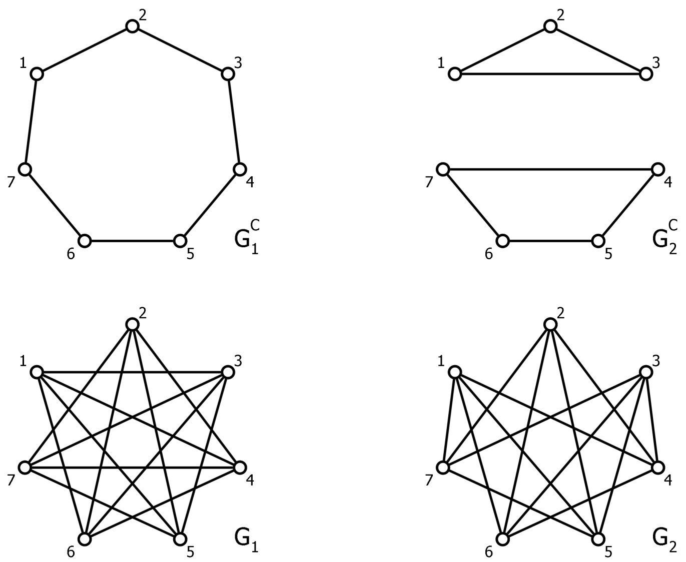
Question 3

(i) Euler's formula states that if a connected plane graph has v vertices, e edges and f faces, then $v + f - e = 2$.
Since G is simple, each vertex has at most one edge to each other vertex, and no loops, so has degree at most $n - 1$. Consequently $k < n$.
The sum of the degrees is nk, and by handshaking, $nk = 2e(G)$, which is even.
Take a plane drawing of G, with f faces. Since G is simple each face has degree at least 3, so the sum of the face degrees is at least $3f$, and by face-handshaking $2e(G) \geq 3f$. Hence $f \leq 2e(G)/3$, and so

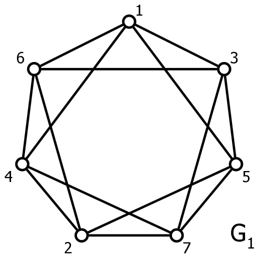
$$\begin{aligned} e(G) &= n + f - 2 \\ &\leq n + 2e(G)/3 - 2 \\ \therefore \qquad e(G)/3 &\leq n - 2 \\ \therefore \qquad e(G) &\leq 3n - 6\,. \end{aligned}$$

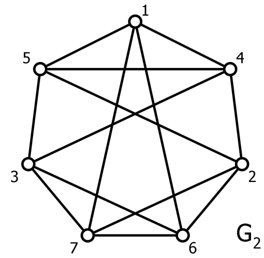
Consequently $nk = 2e(G) \leq 6n - 12$.

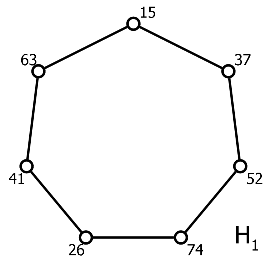
(ii) G^{C} must be a 2-regular simple graph on 7 vertices. There are two possibilities, shown below.

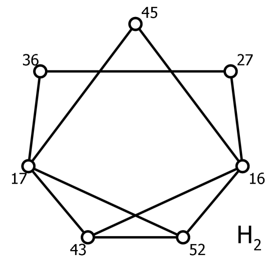


(iii) G_1 has the Hamilton cycle 13572461. We draw G_1 such that this Hamilton cycle forms a regular heptagon, with the other edges being diagonals. We draw the incompatibility graph H_1 which has a vertex for each diagonal drawn, with two vertices of H_1 being adjacent if the corresponding diagonals cross. G_2 has the Hamilton cycle 14267351, and we define H_2 similarly.



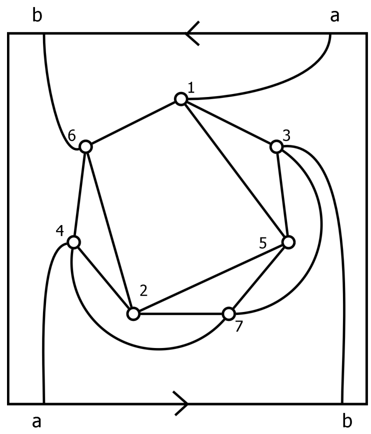


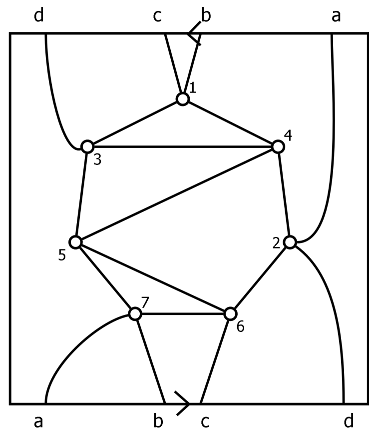




Neither H_1 nor H_2 is bipartite, so neither G_1 nor G_2 is planar.

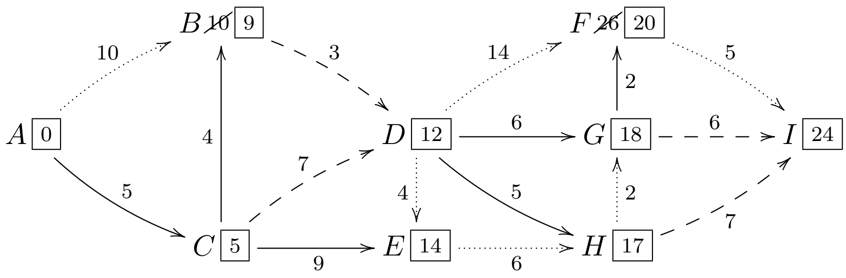
(iv) G_1 and G_2 may be drawn on the Möbius strip as shown (the question only asks for one of these).





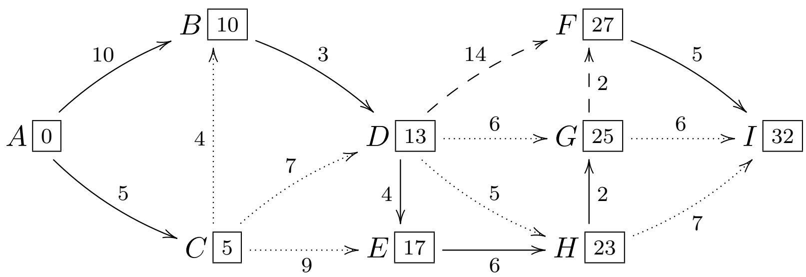
Question 4

(i)(a) Applying Dijkstra's algorithm gives the following. Solid lines are used for the shortest paths; dashed lines indicate a choice of shortest path; dotted lines are not used.



So $s = 24$ and there are four shortest paths: ACDGI, ACDHI, ACBDGI and ACBDHI.

Applying the longest path algorithm gives the following.



So $l = 32$ and there are two longest paths: ABDFI and ABDEHGFI.

(i)(b) Such an arc must be on every shortest path; the only such arc is AC.

(i)(c) Such an arc must be on every longest path and at least one shortest path. There is a unique arc with this property, which is BD.

(ii)(a) Starting from A, the shortest edge is AC, so our first partial tour is just A⇄C. The shortest edge from either of these to a new vertex is CF, so add F after C to get ACFA. The shortest edge from one of these to a new vertex is FE, so add E after F to get ACFEA. Next the shortest edge to a new vertex is EB, giving ACFEBA, then FG, giving ACFGEBA, and then GD, so our final tour is

$$\begin{tikzcd} A \arrow[r, "14"] & C \arrow[r, "13"] & F \arrow[r, "13"] & G \arrow[dl, "17"] \\ B \arrow[u, "18"] & E \arrow[l, "11"'] & D \arrow[l, "21"'] & \end{tikzcd}$$

of total length 107.

(ii)(b) Remove C and use Kruskal's algorithm to find a minimum-weight spanning tree in the remaining vertices. The shortest edges are

BE	EF	CF	FG	DG	BC	$\cdots$
11	12	13	13	17	19	$\cdots$

We can add the first five of these without creating a cycle, so a minimal spanning tree is

$$\begin{tikzcd} B \arrow[r, dash, "11"] & E \arrow[r, dash, "12"] & F \arrow[r, dash, "13"] \arrow[d, dash, "13"] & C \\ & & G \arrow[r, dash, "17"] & D \end{tikzcd}$$

of total length 66.

We then add the two shortest edges from A, AC and AE, with lengths 14 and 17, to get a lower bound of 97.

b) Removing G, the shortest remaining edges are

BE	EF	CF	AC	AE	AB	BC	DE	$\cdots$
11	12	13	14	17	18	19	21	$\cdots$

so we add the first four, skip AE, AB and BC as any of those would create a cycle, and add DE to get

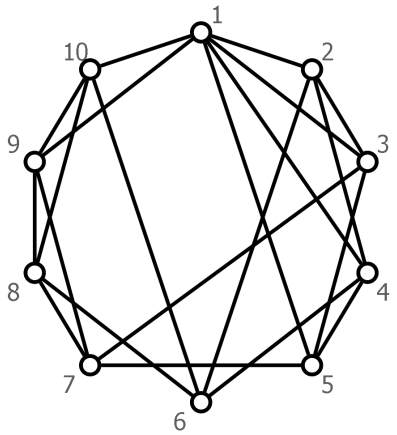
$$\begin{tikzcd} B \arrow[r, dash, "11"] & E \arrow[r, dash, "12"] \arrow[d, dash, "21"] & F \arrow[r, dash, "13"] & C \arrow[d, dash, "14"] \\ & D & & A \end{tikzcd}$$

of total length 71.

We then add the two shortest edges from G, GF and GD, with lengths 13 and 17, to get a lower bound of 101. This is a better lower bound as it is closer to the upper bound.

Question 5

(i) We draw a graph with the exams as vertices, two exams being adjacent if there is some student who has to take both. An assignment of days to exams is just a vertex colouring of this graph.

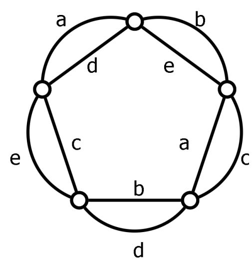


Suppose it can be coloured with three colours, a, b, c. 1, 2 and 3 are all adjacent, so must be different colours; say they are a, b and c in that order. Then 4 is adjacent to 1 and 2, so must be c. Similarly 5 must be b, 6 must be a and 7 must be a. Now 8, 9 and 10 must all be different colours, but none of them can be a. So no such colouring exists.

Thus we need four days, and one such arrangement is to have 1, 6 and 7 on Monday; 2, 5 and 8 on Tuesday; 3 and 9 on Wednesday; and 5 and 10 on Thursday.

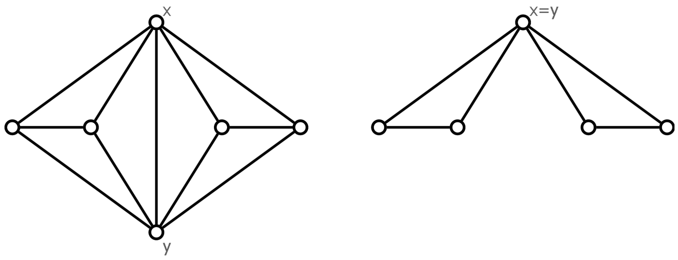
(ii) The chromatic index of a graph is the smallest number of colours needed for an edge-colouring of the graph (where two edges which meet at a vertex must have different colours).

Suppose this graph can be edge-coloured with four colours, a, b, c, d. Each pair of edges between the same two vertices must get different colours. If one pair uses the colours a and b, the next pair must use c and d, the next a and b, the next c and d, and no colours are available for the final pair. So at least five colours are required; a colouring with five colours is shown.



(iii)(a) The chromatic polynomial P_G of a graph G is the function from $\mathbb{Z}^+$ to $\mathbb{N}$ where $P_G(k)$ is the number of vertex colourings of G using colours from the set $\{1, \ldots, k\}$. The chromatic number of G is the smallest k for which $P_G(k) > 0$.

(iii)(b) If G is a simple graph and x and y are non-adjacent vertices then $P_G(k) = P_{G+xy}(k) + P_{G_{x=y}}(k)$, where $G+xy$ is the graph obtained by adding an edge between x and y, and $G_{x=y}$ is that obtained by identifying x and y. Applying these relations to the graph given, with x and y being the top and bottom vertices, results in the following graphs.



$P_{K_r}(k) = k(k-1)\cdots(k-r+1)$. If H consists of graphs H_1 and H_2 glued along an edge then $P_H(k) = \frac{1}{k(k-1)} P_{H_1}(k) P_{H_2}(k)$. $G + xy$ is two copies of K_4 glued along an edge, so

$$\begin{aligned} P_{G+xy}(k) &= \frac{1}{k(k-1)} P_{K_4}(k)^2 \\ &= k(k-1)(k-2)^2(k-3)^2 \,. \end{aligned}$$

Similarly, if H consists of graphs H_1 and H_2 glued at a vertex then $P_H(k) = \frac{1}{k} P_{H_1}(k) P_{H_2}(k)$. $G_{x=y}$ is two copies of K_3 glued at a vertex, so

$$\begin{aligned} P_{G_{x=y}}(k) &= \frac{1}{k} P_{K_3}(k)^2 \\ &= k(k-1)^2(k-2)^2 \,. \end{aligned}$$

Finally,

$$\begin{aligned} P_G(k) &= P_{G+xy}(k) + P_{G_{x=y}}(k) \\ &= k(k-1)(k-2)^2(k-3)^2 + k(k-1)^2(k-2)^2 \\ &= k(k-1)(k-2)^2\big((k-3)^2 + (k-1)\big) \\ &= k(k-1)(k-2)^2\big(k^2 - 5k + 8\big) \,. \end{aligned}$$