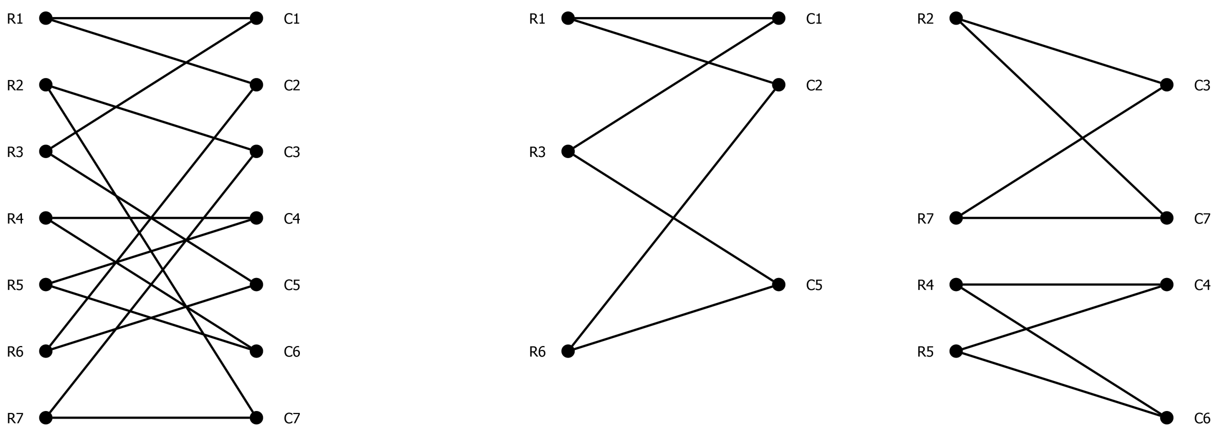
MAS341 Graph Theory
2015 exam solutions

Question 1

(i)(a) Draw a graph with a vertex for each row and column of the framework; connect a row vertex to a column vertex if there is a brace where the row and column meet. We get the graph shown below left.



The graph is disconnected, as shown to the right. So the bracing is not rigid. To make it rigid, we must add enough extra edges to connect the graph, and since there are three components, two braces are sufficient (eg R2C2 and R5C5).

(i)(b) A minimum bracing corresponds to a tree in the row-column graph. This has 9 vertices, so 8 edges. So we need to count rigid bracings with 8 braces; in order to be rigid we must have no empty columns. So there must be one column with two braces (7 choices for this), and every other column has one brace (2 choices for each other column). So there are $7 \times 2^6 = 448$ ways.

(ii) The molecule has 26 vertices, so if it is a tree it must have 25 edges and the sum of the degrees must be 50. Writing x for the valency of X, $6 \times 4 + 17 \times 1 + 3x = 50$, giving $x = 3$.

(iii) An isomorphism between graphs G and H is a bijection ϕ between the vertices of G and the vertices of H, such that the number of edges (of H) between $\phi(v)$ and $\phi(w)$ is the same as the number of edges (of G) between v and w.

Here Graph 4 is not isomorphic to any of the others, as it has 13 edges and they each have 12. Graph 2 is not isomorphic to Graph 1 or Graph 3, as it is bipartite (with a, c, e, g in one class and b, d, f, h in the other), but they have 5-cycles *abcde* and *abcge* respectively. Graph 1 and Graph 3 are isomorphic; an isomorphism is given by the following mapping.

Graph 1	a	b	c	d	e	f	g	h
Graph 3	a	b	f	h	d	c	g	e

Question 2

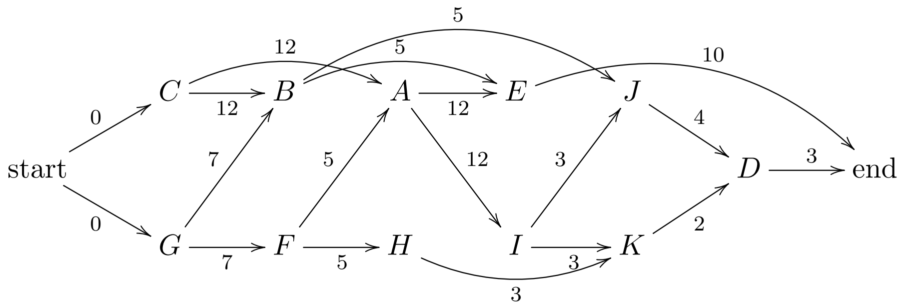
(i)(a) We add edges in the following order, giving the following partial tours:

Edge	Tour
AC	ACA
CB	ACBA
BD	ACBDA
CE	ACEBDA
EG	ACEGBDA
GF	ACEGFBDA

The final tour has length 62+59+40+42+83+47+66=399.

(i)(b) We add edges EG, GF, DB, BC to our spanning tree, reject DC as it would create a cycle, and add CE. This has length 40+42+47+57+59=245. Now we add the two shortest lengths from A, 62+66, to get a lower bound of 373.

(ii)(a) First sort the activities into columns so that each column contains the activities which only depend on activities in previous columns, adding an arc for each precedence with weight equal to the time taken for the earlier activity.



Now run the longest path algorithm on this network to get the earliest start times. The shortest possible time is 34 days.

A	B	C	D	E	F	G	H	I	J	K	end
12	12	0	31	24	7	0	12	24	27	27	34

Now work backwards from end to get the latest start times.

A	B	C	D	E	F	G	H	I	J	K	end
12	19	0	31	24	7	0	26	24	27	29	34

(ii)(b) Such a task must be on every critical path. The critical paths are CAE, CAIJD, GFAE and GFAIJD, so A is the only such task.

(ii)(c) If the project is to be completed in 34 days, task E must start at time 24, and therefore end at time 34, and J must start at time 27 and finish at time 31 (earliest and latest times for these tasks are equal, so they must be started at exactly those times). These two tasks must be worked on simultaneously between time 27 and 31. K must start between time 27 and time 29 inclusive, but this would require three tasks to be performed simultaneously.

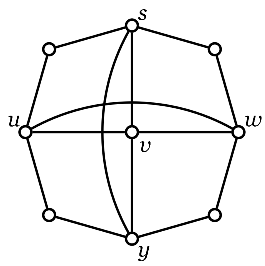
Question 3

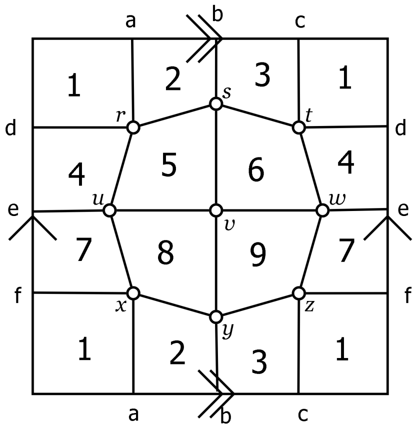
(i) The edges of G may be partitioned into cycles if and only if all degrees of G are even.

Suppose G is a plane graph with all degrees even. Take a set of cycles which partition the edges of G, and colour a face red if it is inside an odd number of these cycles, blue otherwise. Now if two faces share an edge, which is on cycle C, then the two faces are on opposite sides of C but the same side of all other cycles, so one is inside an odd number of cycles and the other is inside an even number. So they get different colours, and this is a face colouring.

(ii)(a) Kuratowski's Theorem states that a graph is non-planar if and only if it contains a subgraph which is a subdivision of K_5 or $K_{3,3}$.

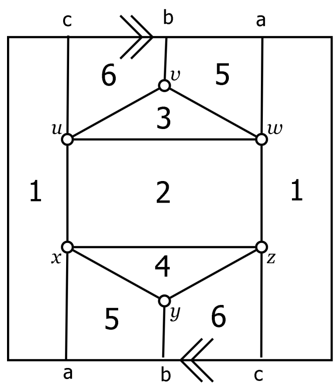
The subgraph shown below left is a subdivision of K_5, so it is not planar (it is possible to find a subdivision of $K_{3,3}$ too). Below right is a drawing on the torus.





(ii)(b) There are 9 faces, numbered in the drawing. The faces 1, 2 and 3 all meet, so three colours are needed. Colouring 1,5,9 red, 2,6,7 blue and 3,4,8 green achieves this.

(iii) Below is a drawing on the Möbius strip. Now faces 1, 5, 6 all meet so need three colours. Colouring 1, 3, 4 red, 2, 5 blue and 6 green will do.



[Note: for parts (ii)(b) and (iii) it is possible to find different drawings which give different answers for the number of colours.]

Question 4

(i)(a) Since each vertex meets one face of degree a, and each such face meets a vertices, the number of faces of degree a is n/a. Since each vertex meets 3 faces of degree b, and each such face meets b vertices, the number of faces of degree b is $3n/b$.

(i)(b) Euler's formula states that if a connected plane graph has v vertices, e edges and f faces, then $v + f - e = 2$. For G, there are n vertices and $n/a + 3n/b$ faces. Since every vertex has degree 4, by handshaking $2e(G) = 4n$, so G has $2n$ edges. Consequently, by Euler's formula, $n + n/a + 3n/b - 2n = 2$, so

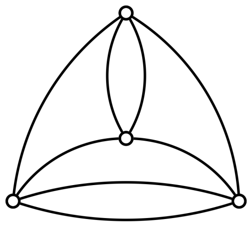
$$\frac{1}{a} + \frac{3}{b} - 1 = \frac{2}{n} > 0\,.$$

(i)(c) G is simple (and has at least 5 vertices, since every vertex has 4 neighbours), so $a, b \geq 3$. *(1 mark)* If $4 \leq a < b$ then $1/a + 3/b < 1/4 + 3/4 = 1$. So we must have $a = 3$. If $b \geq 5$ then $1/a + 3/b \leq 1/3 + 3/5 = 14/15$. So we must have $b = 4$. Now

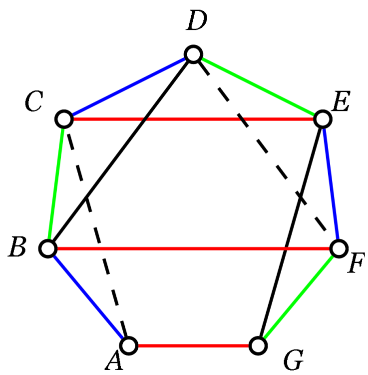
$$\begin{aligned} n &= \frac{2}{\frac{1}{a} + \frac{3}{b} - 1} \\ &= \frac{2}{\frac{1}{3} + \frac{3}{4} - 1} \\ &= 2/\tfrac{1}{12} = 24\,. \end{aligned}$$

Consequently $e(G) = 2n = 48$ and G has $24/3 + 3 \times 24/4 = 26$ faces.

(i)(d) The graph shown works, with $c = 2$ and $d = 3$.



(ii) Draw a graph with teams as vertices and an edge between each pair of teams who must play. The minimum number of weeks needed is the chromatic index of the graph. In an edge colouring, since there are 7 vertices, there can be at most 3 edges of any colour. So at most 12 edges can be coloured with 4 colours, so 5 colours are required.



The colouring shown uses 5 colours, and corresponds to the following schedule.

Week	Matches		
1	A vs G	B vs F	G vs E
2	A vs B	C vs D	E vs F
3	B vs C	D vs E	F vs G
4	A vs C	D vs F	
5	B vs D	E vs G	

Question 5

(i) $G_{x=y}$ is the simple graph obtained by removing x and y and adding a single new vertex, with one edge to every vertex which was adjacent to x or y (or both). The chromatic polynomials satisfy

$$P_G(k) = P_{G_{x=y}}(k) + P_{G+xy}(k)\,,$$

since a k-colouring of $G_{x=y}$ corresponds to a k-colouring of G in which x and y have the same colour, and a k-colouring of $G + xy$ corresponds to a k-colouring of G in which x and y have different colours, so every k-colouring of G is counted exactly once on the right-hand side.

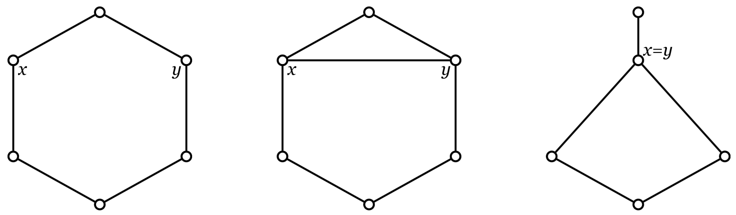
(ii) $P_{C_3}(k) = k(k-1)(k-2)$. Let x and y be non-adjacent vertices of C_4. $C_4 + xy$ is the graph consisting of two copies of C_3 glued along an edge, so

$$\begin{aligned} P_{C_4+xy}(k) &= \frac{1}{k(k-1)}\big(P_{C_3}(k)\big)^2 \\ &= k(k-1)(k-2)^2\,. \end{aligned}$$

$\big(C_4\big)_{x=y}$ is a 3-vertex path. Since it is a tree, it has chromatic polynomial $k(k-1)^2$. So

$$\begin{aligned} P_{C_4}(k) &= P_{(C_4)_{x=y}}(k) + P_{C_4+xy}(k) \\ &= k(k-1)^2 + k(k-1)(k-2)^2 \\ &= k(k-1)(k^2-3k+3)\,. \end{aligned}$$

(iii) If $n > 4$ then let x and y be two vertices of C_n at distance 2. (Illustrated for $n = 6$.)



Now $C_n + xy$ consists of C_3 and C_{n-1} glued along an edge, and so

$$\begin{aligned} P_{C_n+xy}(k) &= \frac{1}{k(k-1)}P_{C_3}(k)P_{C_{n-1}}(k) \\ &= (k-2)P_{C_{n-1}}(k)\,. \end{aligned}$$

The graph $\big(C_n\big)_{x=y}$ consists of C_{n-2} and K_2 glued together at a vertex, so

$$\begin{aligned} P_{(C_n)_{x=y}}(k) &= \frac{1}{k}P_{K_2}(k)P_{C_{n-2}}(k) \\ &= (k-1)P_{C_{n-2}}(k)\,. \end{aligned}$$

Consequently,

$$\begin{aligned} P_{C_n}(k) &= P_{C_n+xy}(k) + P_{(C_n)_{x=y}}(k) \\ &= (k-2)P_{C_{n-1}}(k) + (k-1)P_{C_{n-2}}(k)\,, \end{aligned}$$

as required.

(iv) For $n = 3$,

$$\begin{aligned}(k-1)^3 - (k-1) &= k^3 - 3k^2 + 2k \\ &= k(k-1)(k-2) \\ &= P_{C_3}(k)\,,\end{aligned}$$

and for $n = 4$,

$$\begin{aligned}(k-1)^4 + (k-1) &= k^4 - 4k^3 + 6k^2 - 3k \\ &= k(k-1)(k^2 - 3k + 3) \\ &= P_{C_4}(k)\,,\end{aligned}$$

so the formula holds for $n = 3, 4$ (note that both base cases are needed). Let $n > 4$, and suppose the formula holds for all m with $3 \le m < n$, so in particular it holds for $m = n-1, n-2$. Now

$$\begin{aligned}P_{C_n}(k) &= (k-2)P_{C_{n-1}}(k) + (k-1)P_{C_{n-2}}(k) \\ &= (k-2)\big((k-1)^{n-1} + (-1)^{n-1}(k-1)\big) + (k-1)\big((k-1)^{n-2} + (-1)^{n-2}(k-1)\big) \\ &= (k-2)(k-1)^{n-1} + (k-1)^{n-1} + (-1)^n(k-1)(2-k) + (-1)^n(k-1)(k-1) \\ &= (k-1)(k-1)^n + (-1)^n(k-1)\,,\end{aligned}$$

so the formula also holds for n. Hence, by induction, the result is proved for all $n \ge 3$.

(v) This graph consists of C_4 glued at a vertex to the graph consisting of C_5 and C_6 glued at an edge. So it has polynomial

$$\frac{1}{k}P_{C_4}(k)\left(\frac{1}{k(k-1)}P_{C_5}(k)P_{C_6}(k)\right) = \frac{\big((k-1)^4 + k - 1\big)\big((k-1)^5 - k + 1\big)\big((k-1)^6 + k - 1\big)}{k^2(k-1)}\,.$$