MAS 341: GRAPH THEORY
2016 EXAM SOLUTIONS

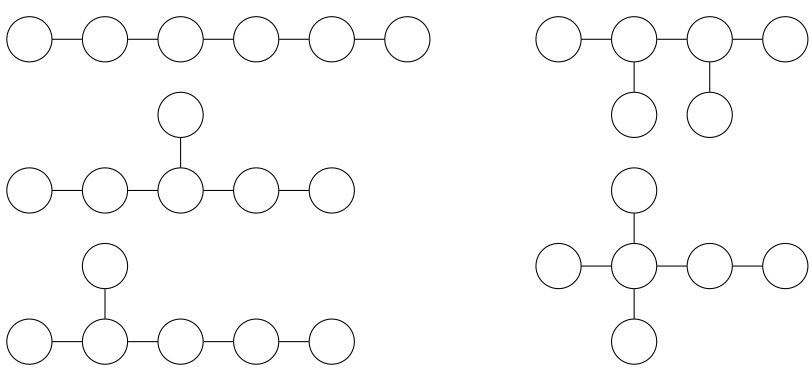
1. Question 1

1.1. Explain why any alkane C_nH_{2n+2} is a tree. How many isomers does C_6H_{14} have? Draw the structure of the carbon atoms in each isomer.

5 marks; 3 marks for proving a tree (1 for connected!), 2 marks for finding and drawing the 5 possibilities.

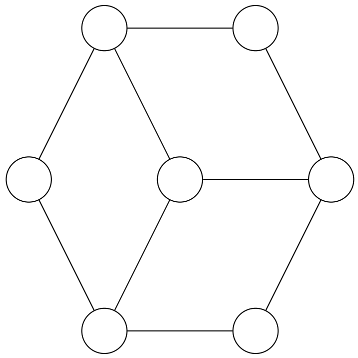
Proof. Carbon has valency 4, and hydrogen has valency 1. C_nH_{2n+2} has $3n+2$ vertices. Using the handshaking lemma, we can calculate the number of edges as half the total degree, which is $4n+2n+2 = 6n+2$. Thus, there are $3n+1$ edges in C_nH_{2n+2}, one less than the number of vertices. Since a molecule is automatically connected, the graph must a tree.

To find all the isomers of C_6H_{14}, draw the carbon atoms – this will be a tree on 6 vertices, with each vertex having degree at most 4. They could be a chain of 6 carbon atoms, a chain of five carbon atoms with 1 other one poking off (from the central vertex of chain, or from next to edge, giving 2 combinations). Finally, it could be a chain of 4 vertices, with the two extra either attached to the same vertex or different. The maximal chain length cannot be 3, as then there would be one central vertex which would have to have degree 5. This gives 5 possibilities, illustrated below:



$\square$

1.2. Consider the graph G given below. Is G Eulerian? Is G Hamiltonian? Is G bipartite? Justify your answers.



6 Marks, 2 for each part, 1 for right answer but incomplete justification.

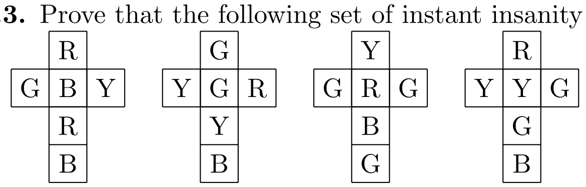
Proof. G is not Eulerian, as it has 4 vertices with odd degree (namely 3).

G is not Hamiltonian – locally near each of the vertices of degree 2, a Hamiltonian path would have to just be forced to be straight through. Together, these give a cycle around the outside of the cube, which misses the interior vertex.

G is bipartite, as it has no odd cycles – color the central vertex and the three vertices of degree two blue, color the other three vertices red.

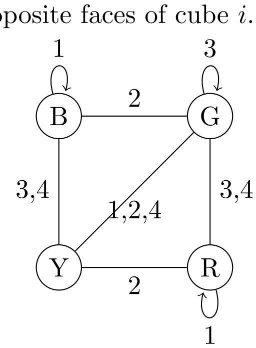
□

1.3. Prove that the following set of instant insanity cubes have no solution.



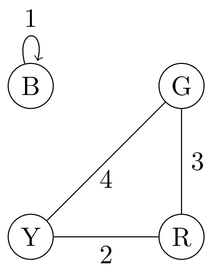
9 marks; 4 for making graph(s) from the cubes, 5 for arguing from this.

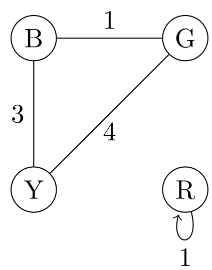
Proof. We make a graph with 4 vertices corresponding to the four colors Blue, Green, Red and Yellow, with an edge labeled i between two colors if they occur on opposite faces of cube i. That gives the following graph:



To select the colors that are on the top/bottom (alternatively, Front/back), we need to find a subgraph that has degree two at every vertex (uses each color twice) and contains one edge with each label 1-4 (uses all the cubes).

There are no four cycles that do this; and no pairs of multiple edges, no sets of four loops, and no pairs of loops + a multiple edge. There are only three cycles + a loop, and only two possibilities, shown below:

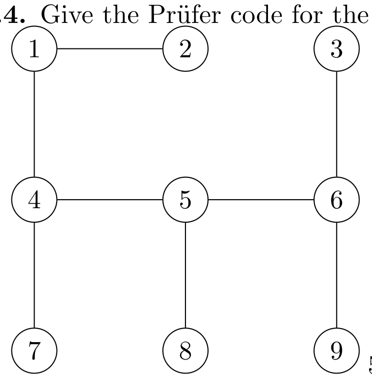




However, the faces appearing on top/bottom and front/back must be disjoint, while both of these solutions contain the edge 4 from G to Y.

□

1.4. Give the Prüfer code for the labelled tree T below.



5 marks; 3 for doing generally the right thing but a minor mistake.

Proof. Continually removing the lowest marked leaf, and recording it and its parent, we obtain

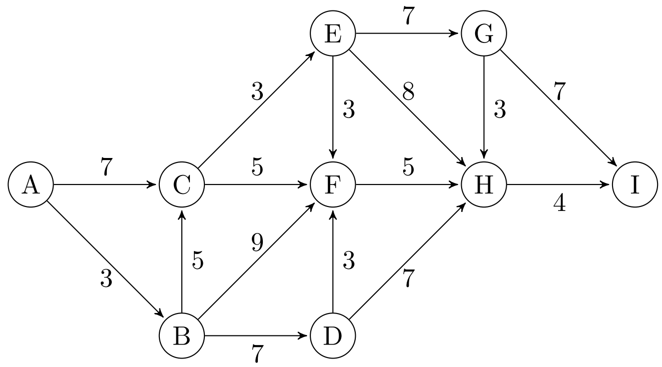
Parent	1	4	6	4	5	5	6	9
Child	2	1	3	7	4	8	5	6

Recording the first $n-2=7$ parent nodes gives the Prüfer code, in this case 1,4,6,4, 5,5,6.

□

2. Question 2

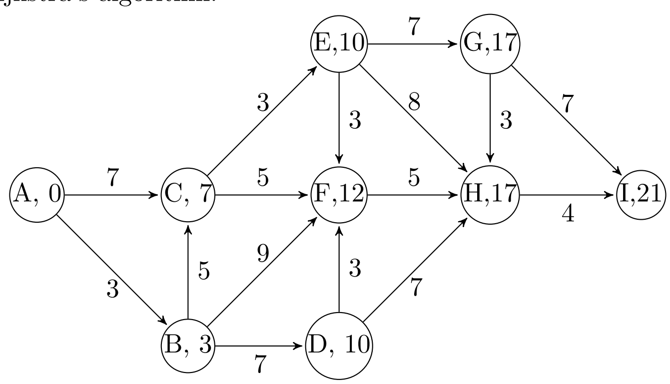
Consider the following directed, weighted graph:



2.1. What is the length s of the shortest path from A to I? For which edges e will shortening e by 0.1 change s? For which edges e will making e longer by 0.1 change s?

8 Marks; 4 for finding length of shortest path and a possible shortest path, 2 for finding all the shortest paths, 1 each for going from there to important edges.

Proof. The shortest path from A to any other point can be solved all at once using Dijkstra's algorithm:



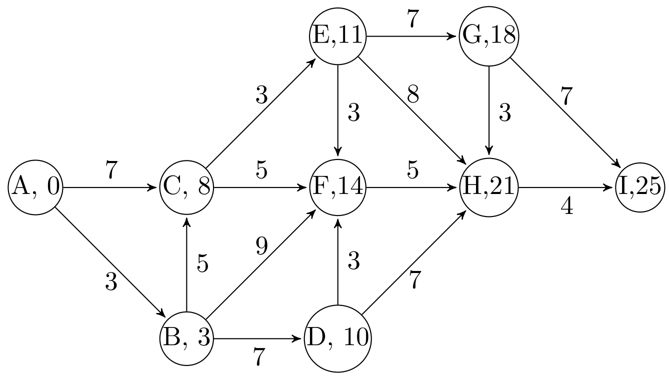
We see the minimum distance from A to I is 21. There are 3 possible such paths: ACFHI, ABFHI, and ABDHI. Increasing an edge length will increase the length of the shortest path if and only if it is in every shortest path – this is only the edge HI. Decreasing the length of an edge will decrease the length of a shortest path if it is in any shortest path, thus, the edges AB, AC, CF, BF, BD, FH, DH, and HI all work.

□

2.2. What is the length ℓ of the longest path from A to I? For which edges e will shortening e by 0.1 change ℓ? For which edges e will making e longer by 0.1 change ℓ?

8 Marks; 4 for finding length of longest path and a possible longest path, 2 for finding all the longest paths, 1 each for going from there to important edges.

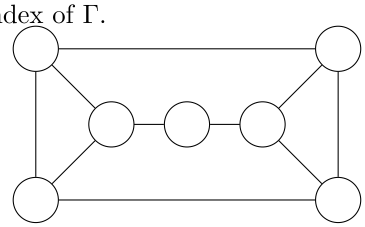
Proof. The longest path algorithm is similar to Dijkstra's algorithm, but depends on the graph being directed acyclic – we first extend to a total order (ABCDEFGHI works), then find the longest path from A to the given vertex one by one. The result is as follows:



We see the maximum path length ℓ is 25, given by two possible paths – both start ABCEGI, or ABCEGHI. Increasing an edge e increases ℓ if and only if e is in *any* longest path, and so the edges AB, BC, CE, EG, GI, GH, and HI would all work. Decreasing an edge e decreases ℓ if and only if e is in *every* longest path, and so only the edges AB, BC, CE, and EG have this result.

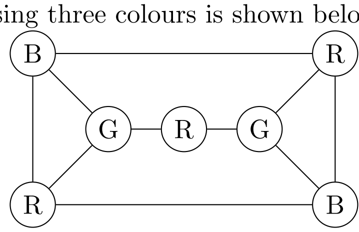
□

2.3. The graph Γ is shown below. Find the chromatic number and the chromatic index of Γ.



5 marks. 2 for chromatic number, split as 1 for colouring, 1 for not bipartite. 3 for chromatic index, split as 1 for colouring, 2 for explaining why 3 isn't possible.

Proof. Since Γ contains triangles, the chromatic number is at least 3. A colouring using three colours is shown below:



The chromatic index is the number of colours needed to colour the edges. Since Γ contains vertices of degree 3, the is at least 3, and is actually equal to 3 or 4 by Vizing's theorem.

For Γ, this number turns out to be four – the two central edges are adjacent, and so must be different colours, say R for the left and and G for the right. Then the two triangles at either end must contain all three colours, with the vertical edges at left and right being R and G, respectively. To be coloured in three colours, the top and bottom edges would then both have to be coloured blue, but some of the diagonal edges of the triangle must be blue, making this not possible. If we colour the top and bottom edges a fourth colour, though, we see the chromatic index is 4.

□

2.4. A finite tree T has at least one vertex v of degree 4, and at least one vertex w of degree 3. Prove that T has at least 5 leaves.

4 Marks.

Proof. We give two proofs. One proof is as follows – since T is connected, there is a path from v and w. Besides this path, there are 3 edges coming out of v, and 2 more edges coming out of w. We build a path starting from each of these edges. If the immediate next vertex is a leaf, we are done. If not, we can continue the path to another edge. Since Γ is a tree, we will never repeat a vertex or join to a vertex contained in one of the other paths, since that would create a loop. Since Γ is finite, the path must eventually terminate, in a leaf.

The second proof uses the handshaking lemma. Suppose T has n vertices, since it is a tree, it must then have $n-1$ edges. Let k_i by the number of vertices of degree i; we have $\sum k_i = n$ since there are n vertices, and $\sum ik_i = 2n-2$ by the handshaking lemma. Subtracting twice the first equation from the second, we ahve $\sum(i-2)k_i = -2$. Only vertices of degree 1, namely leaves, contribute negatively to the sum; the vertex v of degree 4 contributes $+2$, and the vertex w of degree 3 contributes $+1$, so there is a positive contribution of at least 3, so the negative contribution must be at least -5, meaning there are at least 5 leaves. □

3. Problem 3

3.1. Weights are given for edges between 7 vertices, labelled $A - G$.

A						
11	B					
17	9	C				
17	12	14	D			
11	17	15	10	E		
16	9	9	10	8	F	
20	10	21	19	8	12	G

Find a minimal weight spanning tree. What is the total weight of this spanning tree?

5 marks, 3 for generally knowing Kruskal's algorithm and trying to do it, 2 for mistakes.

Proof. We use Kruskal's algorithm, continually adding the lowest weight edge that doesn't create a loop. There are two edges of weight 8, EF and EG. The three edges of weight 9 BC, BF, and CF, create a triangle, and so two of them are added. The two edges of weight 10 both connect D to an edge already in the large component (E or F), and so exactly one of them gets added; similarly, the two edges of weight 11 each connect A to an edge in the main graph, and so one of them is added. This gives a total weight of 16+18+10+11=55. □

3.2. In total, how many spanning trees have the same minimum weight?

4 marks, 2 for general idea.

Proof. The only choice in spanning tree is when edges have tied weight. We had to choose one of the three edges of weight 9 not to add, one of the two edges of weight

10 not to add, and one of the two edges of weight 11 not to add, for 3*2*2=12 total choices. □

3.3. Now, suppose the vertices represent towns, and the weights represent the cost of traveling between towns. A traveling salesperson lives in an 8th town, H; the cost of traveling from H to any town other is 25. The traveling salesperson wants to start at H, travel to every town $A - G$ exactly once, and then return to H, as cheaply as possible. Using your result from the previous part, give a lower bound on the cost of the traveling salesperson's trip. Is this lower bound attainable? Explain.

5 marks; 3 for the lower bound, 2 for explaining why it is not attainable.

Proof. Deleting the vertex H from a solution to the TSP gives a path through the remaining vertices, and in particular a spanning tree. The minimal weight spanning tree has weight 55 from Part (i), and hence any solution to the TSP must have weight at least 55+25+25=105.

This minimum is attained if and only if there is a spanning tree that is a path; however, looking through the 12 possibilities in part 2 we see none of them are a path – the vertices A, D and G have degree 1 in any of the minimal spanning trees. □

3.4. Using the nearest neighbour heuristic starting at H and traveling first to G, give an upper bound on the cost of the cheapest trip for the traveling salesperson.

3 marks, 2 for general idea of heuristic.

Proof. The cost of $H - G$ is 25. From G and continually travel to the cheapest city we haven't yet been to. G-E-F is forced at cost 16. We can then go BCDAH for cost 9+9+14+17+25=74, and total cost 25+16+74=115, or we could go CBADH for cost 9+9+11+17+25=71, and total cost 112. □

3.5. Draw the Petersen graph, and prove it is not Hamiltonian. (Hint, suppose it was Hamiltonian, and consider the edges *not* in the Hamiltonian cycle).

8 marks; 3 for setting up hint and each vertex needing an extra edge, 2 for what no loops, triangles, four cycles tells us, 3 for finishing.

Proof. Drawing not included; see proof in lecture notes.

The Petersen graph has 10 vertices and 15 edges, and every vertex has degree 3. Suppose it was Hamiltonian, with Hamiltonian cycle $v_0 - v_1 - v_2 - \cdots - v_8 - v_9$. This uses 10 edges, there are 5 edges left, with every vertex being included on exactly one.

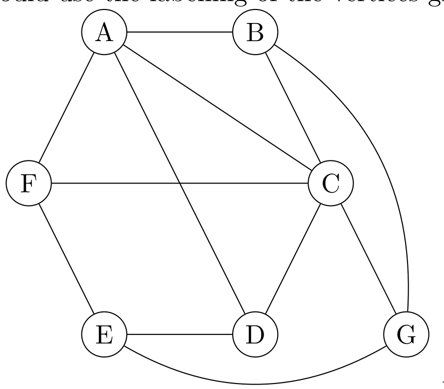
We consider, WLOG, the extra edge adjacent to v_0; it can't be a loop, or be an edge to v_1 or v_2, as the Petersen graph is simple. It can't be to v_2 or v_8 as the Petersen graph has no triangles, and it can't be to v_3 or v_7 as the Petersen graph has no 4 cycles. Thus, the only possibilities are v_4, v_5 or v_6; similarly, there are 3 possible choices for the extra edge at any vertex.

We first prove that not all edges can connect opposite edges: if they did, there would be 4 cycles (e.g., $v_0 - v_1 - v_6 - v_5 - v_0$. So, there is at least one vertex not connected to its opposite vertex, say $v_0 - v_4$. We now consider what v_5 is connected to: it can't be v_0, as that would be make v_0 have degree 4. But connecting it to either v_9 or v_1 makes a 4 cycle.

□

4. Question 4

4.1. State Kuratowski's theorem, and use it to show that the graph G below is not planar. Draw G on the projective plane without edges crossing. Your drawing should use the labelling of the vertices given.

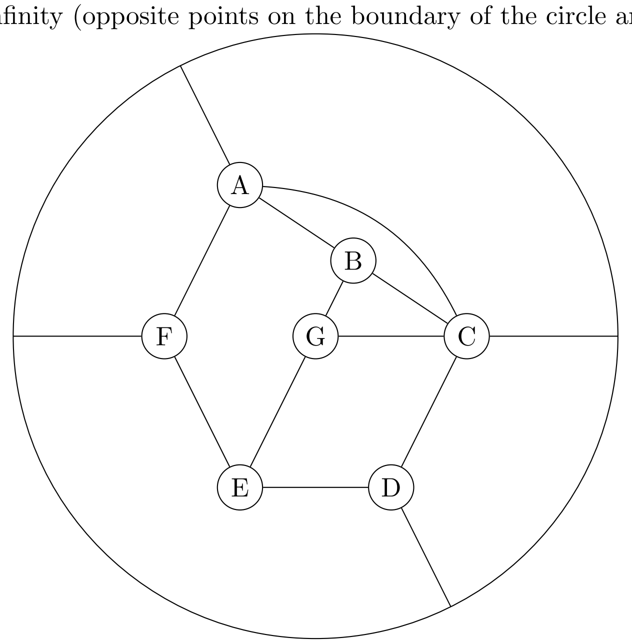


10 Marks, two for stating the theorem, 4 each for applying it and drawing the graph.

Proof. Kuratowski's theorem states that a graph is not planar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

To use Kuratowski's theorem for this graph, we note that the six vertices $A - F$ in the hexagon are almost a $K_{3,3}$ with vertices A, C, E one colour and B, D, F the other colour. The only edge missing is from B to E, but taking $B - G - E$ gives this edge.

To draw G on the projective plane without edges crossing, we turn the hexagon "inside out", so that the one edge crossing between $A - D$ and $F - C$ occurs at infinity (opposite points on the boundary of the circle are identified):



□

4.2. Define the Euler characteristic $\chi(S)$ of a closed, compact surface S and prove it is well defined. Use your drawing of G to calculate the Euler characteristic of the projective plane.

11 marks, 3 for defining Euler characteristic (need edges don't cross, that faces are disks), 6 for proving, and 2 using their graph to calculate the euler characteristic.

Proof. to define the Euler characteristic, suppose that a graph G with v vertices and e edges is drawn on S so that that the edges don't cross and $S \setminus G$ consists of f "faces", each isomorphic to a disk. Then the Euler charactierstic $\chi(S) = v - e + f$.

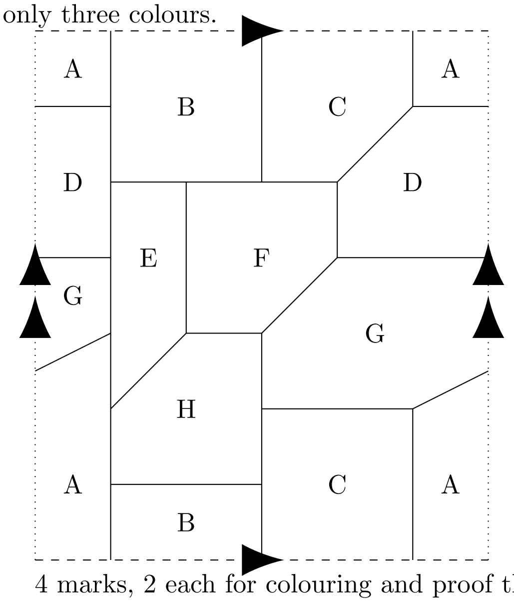
To see this is well defined, suppose that G_1 and G_2 are two graphs embedded this way on S, with v_i vertices, e_i edges, and f_i faces, then we have $v_1 - e_1 + f_1 = v_2 - e_2 + f_2$. To prove this statement, we begin with the observation from topology that any two graphs have a common refinement H, and that G_i can be transformed into H by a sequence of basic moves: removing/adding an edge joining two distinct faces, and, and removing/adding vertices of degree two along an edge.

The key point is that both basic movies keep $v - e + f$ unchanged: the first move decreases e and f by one each, and keeps v unchanged, while the second move decreases e and v by one each and keeps f unchanged.

The Euler characteristic of the projective plane is 1 – our drawing of G has 6 faces – $ABGEF$, ABC, BCG, $CDEG$, $ACFED$ and $CDAF$. G has 7 vertices and 12 edges, and so $\chi = 7 - 12 + 6 = 1$.

□

4.3. Consider the graph Γ drawn below on the torus, with its faces labeled A through H. Give a colouring of the faces of Γ with four colours so that faces meeting along an edge have different colours. Prove that no such colouring is possible with only three colours.



4 marks, 2 each for colouring and proof that 3 isn't possible.

Proof. Most pairs of faces share an edge, and hence must be different colours. Listing the exceptions, we see A and F are not adjacent, and hence both can be coloured blue, B and G are not adjacent and hence both can be coloured green, C and E are not adjacent and hence both can be coloured red, and D and H are not adjacent and hence both can be coloured yellow.

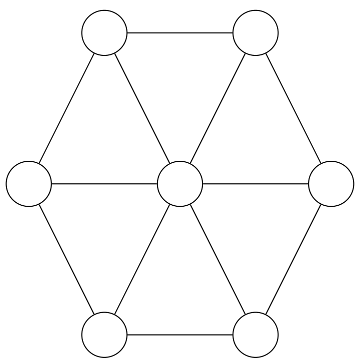
To see that no colouring with only three colours are possible, consider the four faces C,D, F, G (or any set of one face from each of the 4 pairs listed above). Each pair of faces in this set share an edge, and hence cannot be the same colour, thus at least 4 colours are necessary.

□

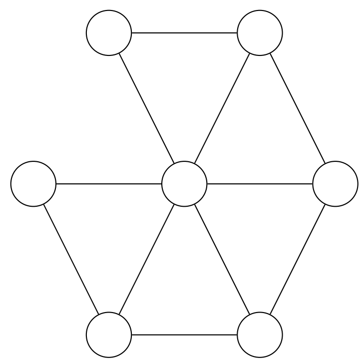
5. Question 5

Recall that the Wheel graph W_n consisted of a copy of the cycle graph C_{n-1}, together with a central vertex v adjacent to every other vertex. We define the *Broken Wheel graph* BW_n, to be the Wheel graph W_n with one edge from the outer cycle removed – we have drawn W_7 and BW_7 below.

W_7



BW_7



5.1. Give the definition of the chromatic polynomial $P_G(k)$. Directly from the definition, prove that the chromatic polynomials of W_n and C_n satisfy the identity:

$$P_{W_n}(k) = kP_{C_{n-1}}(k-1)$$

5 Marks, 2 for definition of chromatic polynomial, 3 for proof.

Proof. $P_{W_n}(k)$ is the number of ways to colour the wheel graph W_n with k colours so that adjacent vertices have different colours. There are k ways to colour the central vertex. The central vertex is adjacent to every other vertex, and so having used one colour for the central vertex, we will only use $k-1$ colours in the rest of the vertices, which exactly form the cycle graph C_{n-1}. Thus, having coloured the central vertex, there are $P_{C_{n-1}}(k-1)$ ways to colour the remaining vertices, and we have $P_{W_n}(k) = kP_{C_{n-1}}(k-1)$.

□

5.2. Prove that

$$P_{BW_n}(k) = k(k-1)(k-2)^{n-2}$$

5 Marks

Proof. Proof 1: begin by colouring the central vertex one of k colours. Go then to one of the vertices on the rim, adjacent to the "missing edge"; this vertex is adjacent to the central vertex, and thus has $k-1$ available colours. We now travel along the central rim, vertex by vertex: each vertex is adjacent to the central vertex, and the previous vertex on the rim we have just coloured, and hence has $k-2$ possible colourings. There are $n-2$ such vertices, and so the total number of possible colourings is $k(k-1)(k-2)^{n-2}$, as desired.

Alternatively, we see that BW_n is obtained from BW_{n-2} by gluing a triangle together on a common edge. This gives $P_{BW_n}(k) = P_{BW_{n-1}}(k)P_{C_3}(k)/[k(k-1)]$. We have $P_{C_3}(k) = k(k-1)(k-2)$, and so this simplifies to $P_{BW_n}(k) = P_{BW_{n-1}}(k)(k-2)$. Using the fact that $BW_3 = C_3$, and induction gives the result. □

5.3. Prove that

$$P_{W_n}(k) = P_{BW_n}(k) - P_{W_{n-1}}(k)$$

Using this, the previous part, and induction, prove that, for $n \geq 4$, we have:

$$P_{W_n} = k(k-2)\left[(k-2)^{n-2} + (-1)^{n-1}\right]$$

10 Marks, 4 for first statement, 6 for second part.

Proof. For G any graph, and e an edge of G, recall that $G \setminus e$ denotes the graph with the edge e removed, and G/e denotes the graph G with an edge removed. We have the following relation on their chromatic polynomials:

$$P_{G\setminus e}(k) = P_G(k) + P_{G/e}(k)$$

where the two terms on the right count colourings of $G \setminus e$ where the two vertices of e get different or the same colours, respectively. Taking G to be W_n, and e to be an edge on the outer cycle, we have $G \setminus e = BW_n$, while $G/e = W_{n-1}$. Substituting these in, and doing a little algebra, gives $P_{W_{n+1}}(k) = P_{BW_n}(k) - W_{n-1}$ as desired.

We prove the second identity by induction. For $n = 4$, we have that $W_4 = K_4$ is the complete graph, which has $P_{K_4}(k) = k(k-1)(k-2)(k-3)$. Plugging $n = 4$ into the right hand side gives $k(k-2)[(k-2)^2 - 1] = k(k-2)(k^2 - 4k + 4 - 1) = k(k-2)(k-3)(k-1)$, proving the base case.

Now, for the inductive hypothesis, assume we have proven that $P_{W_n}(k) = k(k-2)[(k-2)^{n-2} + (-1)^{n-1}]$, and consider $P_{W_{n+1}}(k)$. We have from the first part of this problem that $P_{W_{n+1}}(k) = P_{BW_{n+1}}(k) - P_{W_n}$. From the previous part, we know $P_{BW_{n+1}}(k) = k(k-1)(k-2)^{n-1}$. From the inductive hypothesis, we know that $P_{W_n}(k) = k(k-2)[k-2]^{n-2} + (-1)^{n-2}$. Putting these together, we have

$$\begin{aligned}
P_{W_{n+1}}(k) &= P_{BW_{n+1}}(k) - P_{W_n}(k) \\
&= k(k-1)(k-2)^{n-1} - k(k-2)\left[(k-2)^{n-2} + (-1)^{n-1}\right] \\
&= k(k-2)\left[(k-1)(k-2)^{n-2} - (k-2)^{n-2} + (-1)^n\right] \\
&= k(k-2)\left[(k-2)^{n-1} + (-1)^n\right]
\end{aligned}$$

completing the proof. □

5.4. A graph G has chromatic polynomial $P_G(k) = k^4 - 4k^3 + 5k^2 - 2k$. How many vertices and edges does G have? Is G bipartite? Justify your answers.

5 Marks, 3 for vertices and edges, 2 for bipartite.

Proof. The degree of the chromatic polynomial is the number of vertices, and so G has 4 vertices.

If G has e edges and n vertices, the coefficient of k^{n-1} in $P_G(k)$ is $-e$, and so G has 4 edges.

Being bipartite is equivalent to G having a colouring with only two colours. By definition, the $P_G(2)$ is the number of colourings of G with 2 colours. We have $P_G(2) = 2^4 - 4 \cdot 2^3 + 5 \cdot 2^2 - 2 \cdot 2 = 16 - 32 + 20 - 4 = 0$, and so G is not bipartite. $\square$