MAS341

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2013–2014**

Graph Theory **2 hours 30 minutes**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

1 (i) (a) Define a *tree*, and state how many edges a tree with n vertices must have. ***(2 marks)***

(b) A leaf of a tree is a vertex of degree 1. For $n > 1$, prove that a tree with n vertices must have at least two leaves. ***(3 marks)***

(c) For $n > 1$, prove that a tree with n vertices which has no vertex of degree 2 must have at least $(n + 2)/2$ leaves. ***(4 marks)***

(d) By considering the possible arrangements of vertices which are not leaves, or otherwise, find all different (i.e. non-isomorphic) trees with 10 vertices which have no vertex of degree 2. ***(8 marks)***

(ii) (a) Draw a spanning tree for the graph shown in Figure 1, and draw the corresponding system of fundamental cycles. ***(2 marks)***

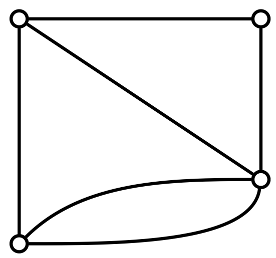


Figure 1

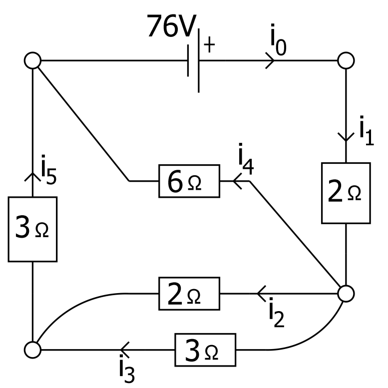


Figure 2

(b) Hence write down six linearly independent equations for the electrical circuit shown in Figure 2, and solve them to find the currents $i_0, \ldots, i_5$. ***(6 marks)***

2 (i) Define *Eulerian*, *semi-Eulerian*, *Hamiltonian* and *semi-Hamiltonian*. Give an example of a graph which is Hamiltonian but not Eulerian. ***(4 marks)***

(ii) Prove that the edges of any graph with all degrees even may be partitioned into disjoint cycles. Must any such graph be Eulerian? ***(8 marks)***

(iii) A group of chess players is scheduled to play a number of games. The schedule is fully determined at the start with games being between two specified players; the same two players may be required to play more than one game against each other. Explain how we may represent this as a graph.

In each game one player plays White and the other plays Black. Suppose every player is involved in an even number of games. Prove that there is a way to arrange the colours so that every player plays exactly half of his or her games as White, and half as Black. Is there always a way to arrange the colours so that no player plays White and Black the same number of times? ***(7 marks)***

(iv) Now suppose there are exactly k players who are each involved in an odd number of games. Explain why k must be even. By adding extra games, or otherwise, prove that there is a way to arrange the colours so that for each player the number of games they play as White is within 1 of the number they play as Black. ***(6 marks)***

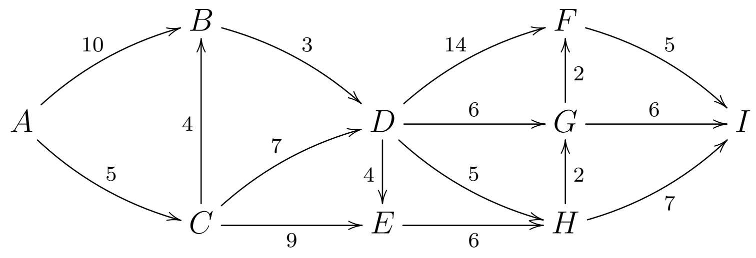
3 (i) State *Euler's formula* for a connected plane graph. Suppose G is a simple connected planar graph with n vertices which is k-regular. Show that $k < n$, that nk is even, and that $nk \leq 6n - 12$. ***(6 marks)***

(ii) Suppose G is a 4-regular simple graph with 7 vertices. What does this imply about G^{c}? Give the two non-isomorphic possibilities for G^{c} and draw the corresponding possibilities for G. ***(4 marks)***

(iii) For each of the two possibilities for G, find a Hamilton cycle and use the planarity algorithm to show that the graph is not planar. You should label the vertices to make it clear what Hamilton cycle you have chosen. ***(10 marks)***

(iv) Choose one of the 4-regular simple graphs with 7 vertices and show how it may be drawn on a Möbius strip without edges crossing. ***(5 marks)***

4 (i) Consider the directed network below.



(a) Use the shortest and longest path algorithms to find all shortest and longest paths from A to I. Give the length s of the shortest path(s) and the length l of the longest path(s). ***(9 marks)***

(b) Which arc cannot be increased in length without changing s? Why is there no other arc with this property? ***(2 marks)***

(c) Is there an arc which cannot be decreased in length without changing both s and l? Justify your answer. ***(3 marks)***

(ii) The distances between seven towns are given in the table below.

A						
18	B					
14	19	C				
25	38	30	D			
17	11	31	21	E		
23	27	13	38	12	F	
19	21	36	17	19	13	G

(a) Use the nearest-insertion heuristic algorithm, starting at A, to find a good upper bound on the travelling salesman problem for these towns. ***(4 marks)***

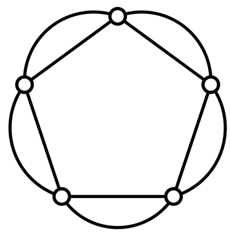
(b) By omitting A, give a good lower bound for the travelling salesman problem. Obtain a second lower bound by omitting G. Which is better? ***(7 marks)***

5 (i) Each of seven students has chosen three courses from ten options, and must sit an exam for each of his or her three choices. Two students sitting the same exam must do so at the same time, but no student can sit more than one exam in the same day. The table of choices is given below.

Student	Exams			Student	Exams		
A	1	2	3	E	6	8	10
B	1	4	5	F	7	8	9
C	2	4	6	G	1	9	10
D	3	5	7				

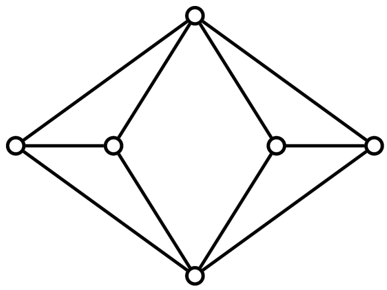
Relate this to an appropriate graph and find the smallest number of days required to schedule the exams. You should give an example of a schedule with that number of days and explain why it cannot be done with fewer. ***(7 marks)***

(ii) Define the *chromatic index* of a graph. What is the chromatic index of the graph shown? ***(4 marks)***



(iii) (a) Define the *chromatic polynomial* of a graph. How can you tell the chromatic number of a graph from its chromatic polynomial? ***(2 marks)***

(b) Find the chromatic polynomial of the graph shown below. You may use standard relations without proof provided they are clearly stated. ***(12 marks)***



End of Question Paper